WIGNER DISTRIBUTION AND ITS APPLICATIONS TO SIGNAL PROCESSING

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CERTIFICATE

Certified that this work entitled WIGNER DISTRIBUTION AND ITS APPLICATIONS TO SIGNAL PROCESSING by Topkar Vijaykumar Anantrao has been carried out under my supervision and that this work has not been submitted elsewhere for a degree

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ABSTRACT

Time-frequency representation of signal and systems is a powerful tool in signal processing specially when the phenomenon under consideration is a nonstationary one. Such a representation has many advantages. It simplifies the analysis, helps in synthesis and design and gives a better insight about the signals, systems and their interaction. Many such representations have been proposed in the past and have been found useful in certain applications. Recently one such distribution 'Wigner Distribution' (WD) has attracted the minds of scientists from various fields like mathematics, optics and signal processing mainly because it has many interesting properties. This thesis tries to study the WD as a member of Cohen's class of time-frequency distributions. Mathematical properties of the WD have been listed and its applications to the fourier optics have been discussed in brief. A new scheme of reception of pulsed chirp signal based on the WD has been proposed and its performance is compared with that of the correlator receiver. The new receiver performs better in detecting the presence of a weak return falling in the side-lobe of a strong return. However the cost of this improvement is paid in terms of the degradation of output SNR.

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CHAPTER 1

INTRODUCTION

In many cases of practical importance mere time domain (signal) or frequency domain (spectrum) representations are not adequate to visualize the phenomenon under investigation and scientists are interested in having a mixed time-frequency representation of signals and systems. Spectrograms used extensively in speech-analysis and the Ambiguity Function used in Radar are some such examples. Such a representation has many advantages It simplifies the analysis of the system, helps in synthesis and design and also gives a better insight about the signals, systems and their interaction. This is particularly so if the phenomenon under consideration is a nonstationary signal which means that at different instants of time the signal has different spectral content.

Through the Fourier Transform, frequency is defined only over an infinite time interval and time is defined only over an infinite frequency band Thus it would appear, at least apparantly, meaningless to ascribe the signal a combined behaviour. But this is not so Each 'line' of the frequency spectrum represents a continuous wave signal of that frequency

and phase and it is to be expected that at times a group of such continuous wave signals will interfere constructively and cause a bunching of signal energy within the spectral band of the group Thus initutively it does appear that energy must have distribution in both time and frequency combined

Many such time-frequency representations have been studied and used in the past and have been found to be useful in certain applications All such representations however can be unified in a unique way Recently one such representation the 'Wigner Distribution' (WD) has been the cynosure of scientists from different fields such as optics, signal processing and mathematics mainly because it exihibits many interesting properties The concept of the WD was introduced in 1932 by the famous physicst E Wigner for describing the probability densities in quantum mechanics [1] It was revived by Ville in 1948 [2] and was used for the first time in signal processing but was again side tracked till it was found to be extremely useful in optics recently [3-7]

For dealing with stationary signals it is possible to estimate the energy content of the signal at different frequencies by taking its Fourier Transform and squaring its

modulus value But this technique fails when the signal is nonstationary For such class of signals what one needs is a representation which reveals how the energy is distributed in time-frequency combined domain An interesting example can be given which brings forth the limitations of the spectral density technique of analysis [8] Consider two signals

$$S_1(t) = \Pi_T(t-T/2)\ \sin\left[2\pi t\ (f_c - aT + at)\right]$$

$$S_2(t) = \frac{\sin \pi B t}{t}\ \cos 2\pi f_c t$$

where B = 2a T defines the bandwidth and f_c the central frequency. $\Pi_T(t - \frac{T}{2})$ indicates rectangular pulse with width T centred at T/2 Fig 1.1 shows their spectral densities which are quite similar This is because the phase information has been lost in this representation. But the fact is that $S_1(t)$ is linearly frequency modulated signal and $S_2(t)$ is not. This clearly shows the need for a tool which will give a better insight into the signal.

The Time-Frequency Distribution (TFD) approach helps in solving such problems Many definitions of TFDs have been proposed with different kernel functions all of which have

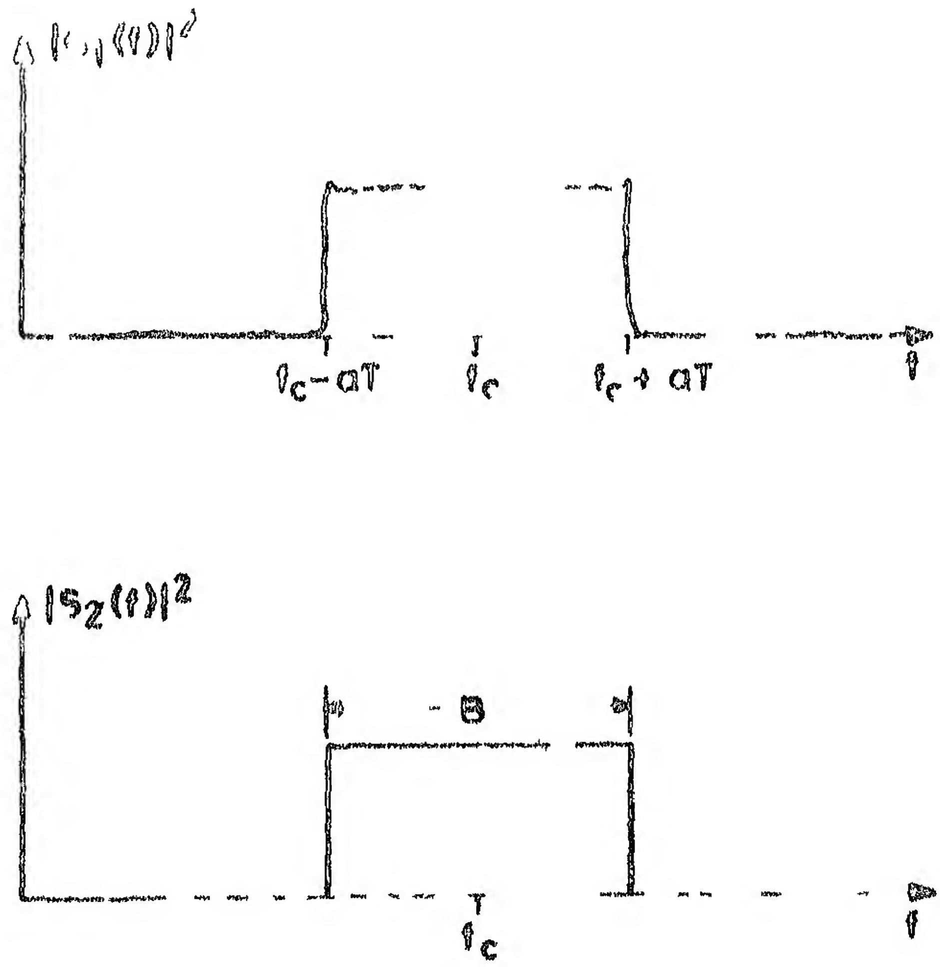


FIG 11

similar mathematical basis although the details differ This thesis tries to study a general class of Time-Frequency Distributions vis a vis detailed study of the WD The outline of the thesis is as follows

Chapter 2 gives a general class of Time-Frequency Distributions with specific examples and their comparisons Chapter 3 deals with mathematical properties of the WD Chapter 4 mentions some of the application areas of the WD in optics Chapter 5 introduces a new scheme of detection of pulsed chirp signals based on the WD and finally chapter 6 includes summary, conclusions and remarks

CHAPTER 2

GENERAL CLASS OF TIME FREQUENCY DISTRIBUTIONS

2 1 INTRODUCTION

As has been mentioned in chapter 1, a mixed time frequency representation or distribution has many advantages in understanding the behaviour of signals and systems as well as in their analysis Many such distributions have been suggested in the past [9] These distributions generally have different properties An efficient way to compare them systematically is to consider a general class of time-frequency distributions that include them all. Section 2 2 gives one such class called Cohen's class of time-frequency distributions Section 2 3 mentions some members of this class and also compares them with each other

2 2 COHEN'S CLASS OF TIME-FREQUENCY DISTRIBUTIONS

A general class of time-frequency distributions was introduced by Cohen [10], [11] and was later widely rewritten by P Flandrin and B Escudie [12],[13]. They showed that all the possible definitions of the time-frequency distributions may be unified in a general definition

$$C_f(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{j(\xi t - \tau\omega - \xi u)}\, \phi(\xi,\tau)\, f(u+\tfrac{\tau}{2})\, f^*(u-\tfrac{\tau}{2})\, du\, d\tau\, d\xi$$

where $f(t)$ is the time signal, $f^*(t)$ is its complex congugate and $\phi(\xi,\tau)$ is a kernel function, representative of the particular distribution

We wish to give a particular distribution an interpretation of energy density over time and frequency This imposes certain constraints on the kernel function $\phi(\xi, t)$ Some other constraints are required to make the distribution meaningful in signal analysis For the purpose of systematic study and comparison of all the distributions belonging to Cohen's class a suitable set of properties was proposed by Classen and Mecklenbraukor [14] These properties and the corresponding constraints on the kernel function are listed in Table 1 [15].

The first two properties are very useful because they enable us to give the distribution an interpretation of energy density over time and frequency. This is because

Table 2.1

Different properties P_i and the corresponding constraints on the kernels $F(\omega)$ is the Fourier transform of the time signal $f(t)$

	Properties	Constraint o Kernel
P1	$\frac{1}{2\pi}\int_{-\infty}^{\infty} C_f(t,\omega)\,d\omega = \lvert f(t)\rvert^2$	$\phi(\xi, 0) = 1$ for all ξ
P2	$\int_{-\infty}^{+\infty} C_f(t,\omega)\,dt = \lvert F(\omega)\rvert^2$	$\phi(0,\tau) = 1$ for all τ
P3	If $g(t) = f(t - t_o)$ then $C_g(t,\omega) = C_f(t - t_o,\omega)$	$\phi(\xi,\tau)$ does not depend on t
P4	If $g(t) = f(t)e^{j\omega_o t}$ then $C_f(t,\omega) = C_f(t,\omega - \omega_o)$	$\phi(\xi,\tau)$ does not depend on ω
P5	$C_f(t,\omega) = C_f^*(t,\omega)$	$\phi(\xi,\tau) = \phi^*(-\xi,-\tau)$
P6	If $f(t) = 0$ for $\lvert t\rvert > T$ then $C_f(t,\omega) = 0$ for $\lvert t\rvert > T$	$\int_{\infty}^{+\infty} e^{j\xi t}\phi(\xi,\tau)\,d\xi = 0$ for $\lvert \tau\rvert < 2\lvert t\rvert$
P7	If $F(\omega) = 0$ for $\lvert\omega\rvert > \Omega$ Then $C_f(t,\omega) = 0$ for $\lvert\omega\rvert > \Omega$	$\int_{-\infty}^{+\infty} e^{-j\omega\tau}\phi(\xi,\tau)\,d\tau = 0$ for $\lvert\Omega\rvert < 2\lvert\omega\rvert$
P8	$\dfrac{\int_{-\infty}^{+\infty} t\,C_f(t,\omega)\,dt}{\int_{-\infty}^{+\infty} C_f(t,\omega)\,dt} = t_g(\omega)$	$\phi(0,\tau) = 1$ for all τ; $\frac{\partial}{\partial\xi}\phi(\xi,\tau)\Big\rvert_{\xi=0} = 0$ for all τ
P9	$\dfrac{\int_{-\infty}^{+\infty} \omega\,C_f(t,\omega)\,d\omega}{\int_{-\infty}^{+\infty} C_f(t,\omega)\,d\omega} = \Omega(t)$	$\phi(\xi,0) = 1$ for all ξ; $\frac{\partial}{\partial\tau}\phi(\xi,\tau)\Big\rvert_{\tau=0} = 0$ for all ξ
P10	$C_f(t,\omega) \geq 0$ for all t and ω	$\phi(\xi,\tau)$ is the ambiguity function of some function $h(t)$

$|f(t)|^2$ is called energy density at time t since

$$\int_{-\infty}^{+\infty} |f(t)|^2 \, dt = \text{Total Energy}$$

Similarly $|F(\omega)|^2$ is called energy spectral density at frequency ω since

$$\int_{-\infty}^{+\infty} |F(\omega)|^2 \, d\omega = \text{Total Energy}$$

Thus $C_f(t, \omega)$ admits of an interpretation as energy density at time t and frequency ω if the integration of $C_f(t,\omega)$ over all frequencies at a fixed time t is the energy density at that time and the integration over all times at a fixed frequency ω is the energy spectral density at that frequency. This in turn guarantees that

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} C_f(t, \omega) \, dt d\omega = \text{Total energy}$$

as required.

Properties P3 and P4 state that shifts in time and frequency give corresponding shifts in the distribution The next property P5, which is very convenient from a practical

point of view, is that the distribution is real valued. The finite support properties P6 and P7 are important They state that if a signal is bounded in time or frequency then the distribution will also be bounded in the same time or frequency band

The next two properties can be very useful for signal analysis Property P8 has the consequence that the centre of gravity or average in the time ~~direction~~ duration at a fixed frequency of the distribution of the impulse response of a linear time-invariant system is equal to the group delay of the system at that frequency The group delay is defined as usual and is equal to

$\frac{-d\phi(\omega)}{d\omega}$ where $\phi(\omega)$ is the phase

Property P9 is a converse statement.

The last property P10 guarantees the positivity of the distribution for all times and frequencies For a meaningful interpretation of $C_f(t,\omega)$ as energy density at time t and frequency ω this property is a must

There has been a lot of discussion regarding the marginality requirement (Properties P1 and P2) and the positivity

requirement (Property P10) of the distributions Wigner has shown that for bilinear distributions it is not possible to satisfy both the requirements [16] This is due to the constraint of Heisenberg's uncertainty principle Although Wigner was quite clear on the bilinearity condition, many subsequent authors have assumed that positive distributions satisfying marginality conditions do not exist at all. This has led to attempts to give some interpretation to the negative values of the distributions and has introduced artificial and false concepts Nonbilinear distributions satisfying both the constraints do exist and have been put forth by many authors [17] Recently Cohen and Posch have given an extremely useful and generalized procedure of generating non-bilinear distributions satisfying both the requirements [18] This in no-way contradicts the Heisenberg's uncertainty principle because the proof of the principle is based only on the marginality property and on the fact that the distribution is bilinear [19]

However till today most of the distributions used commonly in signal processing are bilinear ones and hence can satisfy only one of these two requirements Spectrogram, defined in the next section and used widely in speech analysis [20], is an example which satisfies positivity condition

but not the marginality condition. In fact it has been shown that spectrogram is the only example of Cohen's class with positive distribution [21]. Ambiguity function, Wigner Distribution, Levin Distribution, all defined in section 2.3, satisfy marginality condition but not the positivity condition. The choice whether a distribution to be used should have the positivity property or the marginality property, depends upon the application in hand. But generally the marginality requirement dominates in most of the distributions used in signal processing and hence negative values of the distribution must be accepted and interpreted. The most common interpretation of the negative values still assumes the distribution value as the energy density at time t and frequency ω in the sense that

$$\int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \int_{\omega-\frac{\Delta\omega}{2}}^{\omega+\frac{\Delta\omega}{2}} C_f(t,\omega)\, dt d\omega$$

gives the energy of the signal in the time interval $t - \frac{\Delta t}{2}$ to $t + \frac{\Delta t}{2}$ and in the frequency band $\omega - \frac{\Delta\omega}{2}$ to $\omega + \frac{\Delta\omega}{2}$. But time interval Δt and the frequency band $\Delta\omega$ cannot both be made arbitrarily small simultaneously. Uncertainty principle

restricts the product $\Delta t \; \Delta\omega$ to a lower bound and it can be shown that when $\Delta t \; \Delta\omega$ exceed this bound, the above double integration always yields a positive value [22] Hence $C_f(t, \omega)$ may have negative values at certain points in the t-ω plance but the total energy over any small area of the t -ω plane is always positive, as long as the area used for computation is greater than some minimum value as given by the uncertainty principle This explaination is intitutively satisfying and removes the embarassment caused by the negative values.

2.3 SOME KNOWN TIME-FREQUENCY DISTRIBUTIONS

Some of the known time-frequency distributions with their kernel functions have been listed in Table 2 [15] This table also includes the properties satisfied by a particular distribution It can be seen from the table that many of these distributions satisfy all the properties except the positivity property Hence to choose between them we need an additional criterion. An important criterion is the spread of the square of the magnitude of $C_f(t, \omega)$ [23] As an example let us compare two distributions, namely Wigner Distribution and real part of the Rihaczek distribution, for

Table 2 2

Some known time-frequency distributions with their kernels and corresponding properties

	$C_f(t,\omega)$	Kernel $\phi(\xi,\tau)$	Properties (Table 1)	References	Remarks
...zek	$f^*(t)F(\omega)e^{j\omega t}$	$e^{j\xi\tau/2}$	P1–P4,P6,P7	[24],[25]	Complex valued
...part of ...aczek	$\mathrm{Re}[f^*(t)F(\omega)e^{j\omega t}]$	$\cos(\frac{1}{2}\xi\tau)$	P1–P9	[26]	
	$\frac{\partial}{\partial t}\lvert F_t^-(\omega)\rvert^2 = \mathrm{Re}[2f(t)F_t^-(\omega)e^{j\omega t}]$	$e^{-j\xi\lvert\tau/2\rvert}$	P1–P6	[27]	$F_t^-(\omega)$ is the running spectrum $F_t^-(\omega)=\int_{-\infty}^{t} f(\tau)e^{-j\omega\tau}d$...
...	$-\frac{\partial}{\partial t}\lvert F_t^+(\omega)\rvert^2 = \mathrm{Re}[2f(t)F_t^-(\omega)e^{j\omega t}]$	$e^{j\xi\lvert\tau/2\rvert}$	P1–P6	[26]	$F_t^+(\omega)$ is the running spectrum $F_t^+(\omega)=\int_{t}^{\infty} f(\tau)e^{-j\omega\tau}d$...
...rogram	$\lvert F_t(\omega)\rvert^2$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} V_w(-t,w)e^{j(\omega\tau-\xi t)}d\omega d\tau$	P3–P5,P10	[14],[28]	$V_w(-t,\omega)$ is the Wigner distribution of the window function $w(t)$ $F_t(\omega)=\int_{-\infty}^{+\infty} e^{-j\omega\tau} f($...$w(\tau-t)d\tau$
...ative spectrum	$\lvert F_t^+(\omega)\rvert^2$	$(\pi\delta(-\xi)-\frac{1}{j\xi})e^{j\xi\lvert\tau/2\rvert}$	P3–P5,P10	[29]	For $F_t^+(\omega)$ see Levin...
...ative ...k spectrum	$\lvert F_t^-(\omega)\rvert^2$	$(\pi\delta(\xi)+\frac{1}{j\xi})e^{-j\xi\lvert\tau/2\rvert}$	P3–P5,P10	[29]	For $F_t^-(\omega)$ see Page
...r	$\int_{-\infty}^{+\infty} e^{-j\omega\tau} f(t+\frac{\tau}{2}) f^*(t-\frac{\tau}{2})d\tau$	1	P1–P9		

a chirp signal f(t) given by

$$f(t) = e^{j\alpha t^2}$$

The instantaneous frequency of this signal is given by $2\alpha t$. The Wigner Distribution of this signal is equal to [15]

$$W_f(t, \omega) = 2\pi\delta(\omega - 2\alpha t)$$

This result is extremely satisfying because it gives nonzero distribution value at time t only for frequency $\omega = 2\alpha t$ which is the instantaneous frequency of f(t) at that time

The Rihaczek distribution can be shown to be equal to

$$C_f(t, \omega) = \sqrt{\frac{\pi}{\alpha}} \exp\left(j \frac{(\omega - 2\alpha t)^2}{4\alpha} - j \frac{\pi}{4}\right)$$

The real part of the Rihaczek distribution in which we are interested is equal to

$$C_f(t, \omega) = \sqrt{\frac{\pi}{\alpha}} \cos\left(\frac{(\omega - 2\alpha t)^2}{4\alpha} - \frac{\pi}{4}\right)$$

This shows a large spread around the line $\omega = 2\alpha t$ A better comparison would be to compare the spread $S(t_o, \omega_o)$ of the square magnitude of $C_f(t, \omega)$ at a point (t_o, ω_o) of the $t-\omega$ plane defined as

$$S^2(t_o, \omega_o) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} ((t-t_o)^2 + (\omega-\omega_o)^2) \left|C_f(t,\omega)\right|^2 dt d\omega \quad .$$

It can be shown that the spread of the square magnitude of the Wigner Distribution is minimal as compared to the spread of the Rihaczek Distribution[23] Hence Wigner Distribution is better of the two in analysis dealing with the chirp signals

CHAPTER 3

WIGNER DISTRIBUTION INTRODUCTION AND PROPERTIES

3.1 DEFINITION OF THE WIGNER DISTRIBUTION

In chapter 2 the definition of the Wigner Distribution (WD) as a member of Cohen's class of distributions, has been given More generally the cross WD of two time signals f(t) and g(t) is defined as

$$W_{f,g}(t,\omega) = \int_{-\infty}^{+\infty} f(t+\eta/2)\, g^{*}(t-\eta/2)\, e^{-j\omega\eta}\, d\eta \qquad (3\ 1)$$

where the " * " denotes complex conjugation

If f(t) = g(t) we get auto-WD of the signal f(t)

Let $F(\omega)$ and $G(\omega)$ be the Fourier Transforms of f(t) and g(t) respectively A similar expression for the cross WD of the two spectra $F(\omega)$ and $G(\omega)$ is given by

$$W_{F,G}(\omega,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\Psi t}\, F(\omega+\Psi/2)\, G^{*}(\omega-\Psi/2)\, d\Psi \qquad (3\ 2)$$

It can be shown that

$$W_{F,G}(\omega,t) = W_{f,g}(t,\omega) \qquad (3\ 3)$$

This illustrates the symmetry between the time and frequency domain definitions of the WD

The term 'Distribution' in the name Wigener Distribution suggests the WD to be similar to a joint probability density for the 'random variables' time and frequency The reason for using probabilistic language (e g distribution, expected value, marginality etc) in discussing the WD or any other time-frequency distribution for that matter, is two - fold There is a historical reason that these distributions were first used in quantum mechanics which is inherently a probabilistic theory Hence the original papers by Wigner, Ville or Cohen contain methods and language of quantum mechanics The theory of the time-frequency distributions as established by subsequent workers has the same mathematical structure as joint distributions in probability theory In particular, the requirements one imposes on the distribution function (marginality, positivity) and the methods one uses to calculate physical quantities, that is, integration over the time-frequency plane with a density function is identical to the method used in joint probability theory Hence one may think the time-frequency distribution as a probability distribution and consider time and frequency to be random variables. Obviously, this is not essential always but is suggestive of

the concept involved and hence very useful

In chapter 2 some of the important properties of the WD have been listed in Table 2 Section 3 2 of this chapter gives an exhaustive list of the properties of the WD Sections 3 3, 3 4 and 3 5 deal with the transformations of the t-ω plane which are invariant to the WD Sections 3 3 and 3 4 describe such linear transformations whereas section 3 5 describes nonlinear transformations Section 3 6 presents some illustrations of the WD.

3 2 PROPERTIES OF THE WIGNER DISTRIBUTION

Before listing the properties of the WD the notation used in the discussion is introduced here

The Fourier Transform of a time signal f(t) is denoted as

$$F(\omega) = F\,T\,(f(t)) = \int_{-\infty}^{+\infty} f(t)\, e^{-j\omega t}\, dt \qquad (3\ 4)$$

The inverse Fourier Transform of the spectrum F(ω) is denoted as

$$f(t) = F\,T^{-1}(F(\omega)) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\, e^{+j\omega t}\, d\omega \qquad (3\ 5)$$

The time signal $f(t)$ and its WD as defined in (3 1) with $g(t) = f(t)$, is denoted as

$$f(t) \longleftrightarrow W_{f,f}(t,\omega) \tag{3 6}$$

The inner product of two time signals $f(t)$ and $g(t)$ is defined as

$$\langle f,g \rangle = \int_{-\infty}^{+\infty} f(t) g^*(t)\, dt \tag{3 7}$$

Similarly the inner product of two spectra $F(\omega)$ and $G(\omega)$ is defined as

$$\langle F,G \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega)\; G^*(\omega)\; d\omega \tag{3 8}$$

The norm of a time signal $f(t)$ is defined as

$$||f|| = \langle f,f \rangle^{\frac{1}{2}} \tag{3 9}$$

and similarly for the spectrum $F(\omega)$

$$||F|| = \langle F,F \rangle^{\frac{1}{2}} \quad . \tag{3.10}$$

Several other operations on signals will be introduced below

the shift operation

$$S_{\tau}f = f(t-\tau) \tag{3 11}$$

the (complex) modulation

$$M_{\Omega}f = e^{j\Omega t}\, f(t) \tag{3 12}$$

differentiation

$$Df = \frac{1}{j}\ \frac{df(t)}{dt} \tag{3 13}$$

multiplication by the running variable

$$Lf = t\ f(t) \tag{3 14}$$

and reversal of the running variable

$$Rf = f(-t) \tag{3 15}$$

Most of the properties given below are extentions of similar properties of the Ambiguity Function [30] These can be proved easily from the definition of the WD and hence the proofs have been omitted Some of the proofs and related discussion however can be found in [31] ,[32],[33] and [14] .

1. For any two time signals f(t) and g(t),

$$W_{f,g}(t,\omega) = W^{*}_{g,f}(t,\omega) \tag{3 16}$$

Hence the auto WD of any (real or complex) signal is always

real Moreover the WD of a real signal is an even function of the frequency, i e

$$W_{f,f}(t,\omega) = W_{f,f}(t, -\omega) \tag{3 17}$$

2 Time Shift A time shift in both the signals f(t) and g(t) corresponds to the same time shift in the WD, i e.

$$W_{S_\tau f,\, S_\tau g}(t,\omega) = W_{f,g}(t-\tau,\, \omega) \tag{3 18}$$

3 Frequency Shift Modulating both the signals f(t) and g(t) by $e^{j\Omega t}$ results in the same frequency shift of the WD, i e

$$W_{M_\Omega f,\, M_\Omega g}(t,\omega) = W_{f,g}(t,\omega-\Omega) \tag{3 19}$$

4 Combining property 2 and property 3 we get

$$W_{M_\Omega S_\tau f,\, M_\Omega S_\tau g}(t,\omega) = W_{f,g}(t-\tau,\, \omega-\Omega) \tag{3.20}$$

This result can be used to express the WD as

$$W_{f,g}(t,\omega) = W_{S_{-t}M_{-\omega} f,\, S_{-t}M_{-\omega} g}(0,0) \tag{3.21}$$

Using the definition of the inner product and the operator R given in (3 7) and (3 15) we get

$$W_{f,g}(0,0) = 2\left\langle f,\, Rg \right\rangle \tag{3 22}$$

From the above two equations it then follows that

$$W_{f,g}(t,\omega) = 2 \left\langle S_{-t} M_{-\omega} f, RS_{-t} M_{-\omega} g \right\rangle \tag{3.23}$$

Thus the value of the WD at a certain time t and frequency ω can be determined by the inner product of the shifted and modulated signal, the second of which has undergone a time reversal

5 Addition of Two Signals The WD of two signals f(t) and g(t) is a bilinear functional of f and g which means that the WD of the sum of the two signals is not simply the sum of the WD's of the individual signals It can be shown that

$$W_{f_1+f_2,\, g_1+g_2}(t,\omega) = W_{f_1,g_1}(t,\omega) + W_{f_1,g_2}(t,\omega) + W_{f_2,g_1}(t,\omega) + W_{f_2,g_2}(t,\omega) \tag{3 24}$$

As a particular case

$$W_{f+g,\, f+g}(t,\omega) = W_{f,f}(t,\omega) + W_{g,g}(t,\omega) + 2 \text{ Real Part } (W_{f,g}(t,\omega)) . \tag{3 25}$$

6 The product of the WD of two signals f(t) and g(t) with running variable t can be expressed as a sum of two WD's

$$t\, W_{f,g}(t,\omega) = \frac{1}{2} W_{Lf,g}(t,\omega) + \frac{1}{2} W_{f,Lg}(t,\omega) \tag{3 26}$$

where the operator L has been defined in (3 14) Similarly the multiplication of the WD by ω can be expressed as

$$\omega W_{f,g}(t,\omega) = \frac{1}{2} W_{Df,g}(t,\omega) + W_{f,Dg}(t,\omega) \qquad (3\ 27)$$

where the operator D is defined in (3 13)

7 According to the definition (3 1), the WD can be treated as the spectrum (or the Fourier Transform) of the signal $f(t+\frac{\tau}{2})g^*(t-\frac{\tau}{2})$ considered as a function of τ with t as a fixed parameter Therefore the inverse Fourier Transform yields

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega\tau} W_{f,g}(t,\omega)\, d\omega = f(t+\frac{\tau}{2})g^*(t-\frac{\tau}{2})$$

which can be written as

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega(t1-t_2)} W_{f,g}\left(\frac{t_1+t_2}{2},\omega\right) d\omega = f(t_1)g^*(t_2) \qquad (3.28)$$

If $t_1=t_2=t$ we get

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} W_{f,g}(t,\omega)\, d\omega = f(t)g^*(t) \qquad (3\ 29)$$

and in particular,

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} W_{f,f}(t,\omega)\, d\omega = |f(t)|^2 \quad . \qquad (3\ 30)$$

This proves the marginality property of the WD mentioned in the Table 2 of chapter 2

If $t_1 = t$ and $t_2 = 0$ we get

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega\tau} W_{f,g}(t/2, \omega)\, d\omega = f(t) g^{*}(0) \qquad (3.31)$$

This is an interesting result which shows that f(t) can be completely recovered from its cross WD at time t/2, upto the constant factor $g^{*}(0)$. Same is true for g(t). A similar discussion applies to the definition of the WD given in (3.2). Here the WD can be considered as the inverse Fourier Transform of the spectrum $F(\omega+\frac{\Psi}{2}) G^{*}(\omega-\frac{\Psi}{2})$ considered as a function of Ψ with ω as fixed parameter. Therefore the Fourier Transform yields

$$\int_{-\infty}^{+\infty} e^{-j\Psi t} W_{F,G}(\omega, t)\, dt = F(\omega+\frac{\Psi}{2}) G^{*}(\omega-\Psi/2)$$

using (3.3) and changing the varibles we get

$$\int_{-\infty}^{+\infty} e^{-j(\omega_1-\omega_2)t} W_{f,g}(t, \frac{\omega_1+\omega_2}{2})\, dt = F(\omega_1) G^{*}(\omega_2) \qquad (3.32)$$

If $\omega_1 = \omega_2$ we get

$$\int_{-\infty}^{+\infty} W_{f,g}(t, \omega)\, dt = F(\omega)\, G^{*}(\omega) \qquad (3.33)$$

and in particular

$$\int_{-\infty}^{+\infty} W_{f,f}(t,\omega)\, dt = |F(\omega)|^2 \tag{3 34}$$

This is another marginality property of the WD

If $\omega_1 = \omega$ and $\omega_2 = 0$ we get

$$\int_{-\infty}^{\infty} e^{-j\omega t}\, W_{f,g}(t, \omega/2)\, dt = F(\omega)G^*(0) \tag{3 35}$$

This relation shows that the spectrum of f(t) can be completely recovered upto the constant $G^*(0)$ from its cross WD at frequency $\omega/2$.

8 From equation (3.29) it follows that

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_{f,g}(t,\omega)\, d\omega dt = \int_{-\infty}^{+\infty} f(t) g^*(t) dt = \langle f,g \rangle \tag{3 36}$$

Similarly from equation (3 30) we get

$$\frac{1}{2\pi} \int_{t_a}^{t_b} \int_{-\infty}^{+\infty} W_{f,f}(t, \omega)\, d\omega\, dt = \int_{t_a}^{t_b} |f(t)|^2\, dt \tag{3 37}$$

Thus the energy of the signal between the time intervals t_a to t_b can be found from its WD Similar results exist

for the spectrums $F(\omega)$ and $G(\omega)$

9 Moyal's Formula Moyal's formula for the integration of the product of two WD's is as follows

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W_{f_1,g_1}(t,\omega)\, W_{f_2,g_2}(t,\omega)\, dt d\omega$$

$$= \left\langle f_1, f_2 \right\rangle \left\langle g_1, g_2 \right\rangle^{*} \qquad (3\ 38)$$

10. Time Limited Signals If $f(t)$ and $g(t)$ are restricted to finite time interval only and

$$f(t) = g(t) = 0 \text{ for } t_a < t < t_b \qquad . (3\ 39)$$

then the WD is restricted to the same interval, i e.

$$W_{f,g}(t, \omega) = 0 \text{ for } t_a < t < t_b, \quad \text{For all } \omega \qquad .(3\ 40)$$

11 Band Limited Signals If $f(t)$ and $g(t)$ are both bands-limited, then the WD is limited to the same band, i e if

$$F(\omega) = G(\omega) = 0 \text{ for } \omega_a > \omega < \omega_b \qquad (3\ 41)$$

then

$$W_{f,g}(t,\omega) = 0 \text{ for } \omega_a < \omega < \omega_b, \qquad \text{For all } t \qquad (3\ 42)$$

12 The WD and the Ambiguity Function As has been mentioned in chapter 2, the Ambiguity Function (AF) of two signals $f(t)$ and $g(t)$ is defined as

$$A_{f,g}(\tau, \sigma) = \int_{-\infty}^{+\infty} f(\eta-\frac{\tau}{2})\, g^*(\eta+\frac{\tau}{2})\, e^{-j\sigma\eta}\, d\eta \qquad (3\ 43)$$

or equivalently

$$A_{fg}(\tau, \sigma) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\varphi+\frac{\sigma}{2})\, G^*(\varphi-\frac{\sigma}{2})\, e^{j\tau\varphi} d\varphi \qquad (3\ 44)$$

It can be shown that the WD and the AF of two signals are related by a two-dimensional Fourier Transform, i e

$$A_{f,g}(\tau, \sigma) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j(\sigma t - \tau\omega)} W_{f,g}(t,\omega)\ dt d\omega \qquad (3\ 45)$$

and

$$W_{f,g}(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j(\omega\tau - t\sigma)} A_{f,g}(\tau, \sigma)\ d\tau d\sigma \qquad . (3\ 46)$$

13 If f(t) and g(t) are both even or both odd functions of time, i.e

$$f(t) = \pm f(-t) \qquad (3\ 47)$$

and $g(t) = \pm g(-t)$, then the AF and the WD are the same upto scale factors, i e.

$$W_{f,g}(t, \omega) = \pm 2\, A_{f,g}(2t, 2\omega) \qquad . (3\ 48)$$

14 All the Time-Frequency Distributions of the Cohen's class can be obtained by a linear transformation of the WD, characterized by the kernel $\varphi(t, \omega)$ which is related to $\Phi(\Psi, \tau)$ by a two-dimensional Fourier Transform, i e.

$$C_f(t, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \varphi(t-\tau, \omega - \Psi) \; W_{f,g}(\tau, \Psi) \, d\tau d\Psi \tag{3 49}$$

$$\text{where } \varphi(t,\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{j(\Psi t - \omega\tau)} \, \Phi(\Psi, \tau) d\Psi d\tau \tag{3 50}$$

15 The auto WDs of two signals f(t) and g(t) are identical iff

$$f(t) = e^{j\lambda} g(t) \tag{3 51}$$

where λ is a real constant

16 If $W_{f,f}(t, \omega) \leftrightarrow f(t)$

then $W_{f,f}(at, \omega/a) \leftrightarrow \sqrt{|a|} \, f(at)$ (3 52)

where a is a constant This means that the time stretching of the signal f(t) affects both the axis of the t-ω plane

17 If $W_{f,f}(t,\omega) \leftrightarrow f(t)$

then $W_{f,f}(t, \omega - 2bt) \leftrightarrow \hat{f}(t) = f(t)e^{jbt^2}$ (3 53)

where b is a real constant

This means that if the WD has most of its energy along the t-axis then by chirping the energy concentration is moved to the line $\omega = 2bt$.

18 If $W_{ff}(t, \omega) \longleftrightarrow f(t)$

then $W_{ff}(t+2d\omega, \omega) \longleftrightarrow \hat{\hat{f}}(t) = F\,T^{-1}(F(\omega)e^{jd\omega^2}))$

(3 54)

where d is a real constant

This means if the WD has most of its energy along the ω axis then by frequency chirping the signal, the energy concentration is moved to the line $t = -2d\omega$.

19 Let $W_{f,f}(t, \omega) \longleftrightarrow f(t)$

$W_{g,g}(t, \omega) \longleftrightarrow g(t)$

and $S(t, \omega) - W_{f,f}(t, \omega) + W_{g,g}(t, \omega)$ (3 55)

then $S(t, \omega)$ is an auto WD iff

$f(t) = k_1 g(t)$. (3 56)

and hence iff

$W_{f,f}(t, \omega) = k_2\, W_{g,g}(t, \omega)$ (3 57)

where k_1 is any constant and $k_2 - |k_1|^2$

21 Invariance under Rotation of t- Plane The rotated WD is still a WD which means that if

$W_{f,f}(t, \omega) \longleftrightarrow f(t)$

then

$W_{f,f}(\omega \cos O + t \sin O, \omega \cos O - t \sin O) \longleftrightarrow g(t)$

(3 58)

where $$g(t) = e^{+j\frac{t^2 \tan\theta}{2}} \frac{1}{\pi\sqrt{2}} \int_{-\infty}^{+\infty} F(\omega)\, e^{j\frac{\omega^2 \tan\theta}{2}}$$

$$e^{j\omega t \sec\theta}\, d\omega \qquad (3\ 59)$$

21 Let $W_{f,f}(t,\omega) \longleftrightarrow f(t)$

$W_{g,g}(t,\omega) \longleftrightarrow g(t)$

$W_{h,h}(t,\omega) \longleftrightarrow h(t)$

a) If $h(t) = f(t).g(t)$ (3 60)

then $$W_{h,h}(t,\omega) = \frac{1}{2\pi} W_{f,f}(t,\omega) \overset{\omega}{*} W_{g,g}(t,\omega) \qquad (3\ 61)$$

where $\overset{\omega}{*}$ indicate convolution w r t ω

b) If $h(t)=u(t) * v(t)$ (3,62)

then $W_{h,h}(t,\omega) = W_{f,f}(t,\omega) \overset{t}{\ } W_{g,g}(t,\omega)$

where $\overset{t}{*}$ indicate convolution w r t t

22 Let u(t), v(t), g(t), h(t) be time signals

Then $$W_{ug,gh}(t,\omega) = \frac{1}{2\pi} W_{ug,ug}(t,\omega) \overset{\omega}{*} W_{vh,vh}(t,\omega) \qquad (3\ 63)$$

or $$= \frac{1}{2\pi} W_{uh,uh}(t,\omega) \overset{\omega}{*} W_{vg,vg}(t,\omega) \qquad (3\ 64)$$

and $W_{u^*v, g^*h}(t, \omega) = W_{ug, ug}(t, \omega) \overset{t}{*} W_{vh, vh}(t, \omega)$ (3 65)

or $= W_{uh, uh}(t, \omega) \overset{t}{*} W_{vg, vg}(t, \omega)$ (3 66)

23 Let $W_{f,f}(t, \omega) \longleftrightarrow f(t)$

$W_{g,g}(t, \omega) \longleftrightarrow g(t)$

Then $W_{f,g}(t, \omega) = \frac{1}{2\pi} W_{f,f}(t, \omega) \overset{\omega}{*} (-2)\ e^{-j2\omega t}$

$(F\ T\ (\frac{g}{f})^*)(-2\omega)$ (3 67)

where $(F\ T\ (\frac{g}{f})^*)(-2\omega)$ indicate Fourier Transform of the time function $g^*(t)/f^*(t)$ at frequency -2ω

24 Realizability Condition of the WD A distribution $C(t, \omega)$ is an auto WD of some time signal iff

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} C(\frac{x+y}{2}, \omega)\ e^{j\omega(x-y)}\ d\omega = f(x) f^*(y) \quad (3\ 68)$$

where $f(t)$ is any time signal and in that case)

$$C(t, \omega) = W_{f,f}(t, \omega) \quad . (3\ 69)$$

Similarly $C(t, \omega)$ is an auto WD of some spectrum iff

$$\int_{-\infty}^{+\infty} C(t, \frac{x+y}{2})\ e^{-jt(x-y)}\ dt = F(x)\ F^*(y) \quad \text{--}(3\ 70)$$

for some spectrum $\Gamma(\omega)$ and in that case,

$$C(t, \omega) - W_{ff}(t, \omega) \tag{3 71}$$

3 3 LINEAR TRANSFORMATION OF THE t-ω PLANE

The WD of a given time signal may have high density areas in certain parts of the t-ω plane If for some reason we wish to shift this area of high energy concentration to some other area, transformation of the t-ω plane is called for But we are interested in only those transformations which are invariant to the WD In other words the transformation should be such that the distribution with transformed coordinates is still a WD corresponding to some time signal Thus if the transformation is invariant to the WD we can obtain a new time signal with desired distribution of energy in t-ω plane from the given signal and its distribution of energy In this and next section we will deal only with linear transformations of the t-ω plane In this section we will assume linear transformation of ω axis only Next section deals with general linear transformation of the t-ω plane. In section 3.5 we will discuss nonlinear transformations All this analysis is an extension of a similar analysis of the Ambiguity Function [30].

Let the transformation of the t-ω plane be as follows

$$t' = t \qquad (3\ 72)$$

$$\omega' = k\omega \qquad (3\ 73)$$

where t' and ω' denote the axes of the transformed plane and k is a real constant

Let us see under what conditions this transformation leads to the invariance of the WD In other words we wish to find the condition on the value of k such that after the transformation of the t-ω plane the modified time-frequency distribution remains a WD corresponding to some time signal

Let $W_{f,f}(t,\omega) \longleftrightarrow f(t)$

and $C(t,\omega) - W_{f,f}(t',\omega') = W_{f,f}(t, k\omega)$. $\qquad (3\ 74)$

From equation (3 70), $C(t,\omega)$ is also an auto WD iff

$$\int_{-\infty}^{+\infty} C(t, \tfrac{x+y}{2})\, e^{-jt(x-y)}\, dt = G(x)\, G^*(y)$$

for some spectrum $G(\omega)$

i e iff $$\int_{-\infty}^{+\infty} W_{f,f}(t, \tfrac{k(x+y)}{2})\, e^{-jt(x-y)}\, dt = G(x) G^*(y)$$

i e iff $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t+\tfrac{\eta}{2})\, f^*(t-\tfrac{\eta}{2})\, e^{-j\frac{k(x+y)\eta}{2}}\, e^{-jt(x-y)}\, d\eta\, dt = G(x) G^*(y)$$

Let $t + \frac{\eta}{2} = p, \quad t - \frac{\eta}{2} = q$

$$t = \frac{p+q}{2}, \quad \eta = p-q$$

$C(t,\omega)$ is an auto WD iff

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(p)\, f^*(q)\, e^{-j\frac{k(x+y)}{2}(p-q)}\, e^{-j(x-y)(p+q)/2}\, dp\,dq$$

$$= G(x)G^*(y)$$

i e iff $F\left[\frac{k(x+y)}{2} + \frac{(x-y)}{2}\right] F^*\left[\frac{k(x+y)}{2} - \frac{(x-y)}{2}\right] = G(x)G^*(y)$

i e iff $F\left[(k+1)x + (k-1)y\right] F^*\left[(k-1)x + (k+1)y\right] = G(2x)G^*(2y)$

for all values of x and y

Putting $x = 0$ we get

$$F\left[(k-1)y\right] F^*\left[(k+1)y\right] = G(0)G^*(2y)$$

Putting $y = 0$ we get

$$F\left[(k+1)x\right] F^*\left[(k-1)x\right] = G(2x)G^*(0)$$

Hence we get

$$G(2x)\, G^*(2y) = \frac{F\left[(k-1)y\right] F^*\left[(k+1)y\right] F\left[(k+1)x\right] F^*\left[(k-1)x\right]}{|G(0)|^2} \qquad (3\ 75)$$

But $|G(0)|^2 = |F(0)|^2$

$G(t, \omega)$ is an auto WD iff

$$F[(k+1)x - (k-1)y] \; F^*[(k-1)x + (k+1)y]$$

$$= \frac{F[(k-1)y] \; F^*[(k+1)y] \; F[(k+1)x] \; F^*[(k-1)x]}{|F(0)|^2}$$

(3 76)

This shows that only those values of k which satisfy equation (3 76) will give rise to an invariant linear transformation of t-ω plane (assuming t' = t) It can be seen from equation (3.76) that $k = \pm 1$ always satisfy this condition, a result as expected.

Special Case

Let $F(\omega)$ be a rational function which is the most important case in practice, i e

$$F(\omega) = \frac{P(\omega)}{Q(\omega)} = \frac{\prod_i (\omega - \omega_i)}{\prod_j (\omega - \omega_j)} \qquad (3\ 77)$$

Let us find conditions under which $G(\omega)$ is also a rational function since this also is the most commonly required condition Now from equation (3 75) with $\omega = x$ and $\lambda = y$ we get

$$G(2\omega) \; G^*(2\lambda) = F[(k+1)\omega + (k-1)\lambda] \; F^*[(k-1)\omega + (k+1)\lambda]$$

(3 78)

Let $G(\omega) = \frac{\prod_m (\omega - \omega_m)}{\prod_1 (\omega - \omega_1)}$

Thus for all $\omega = \omega_m/2$ L H S of the equation (3 78) above is zero for all values of λ, except at $\lambda = \omega_1/2$

$$F\left[(k+1)\,\omega_m/2 + (k-1)\lambda\right] = 0 \qquad (3\ 79)$$

$$\text{or } F^*\left[(k-1)\,\omega_m/2 + (k+1)\lambda\right] = 0 \qquad (3\ 80)$$

for all λ's except $\lambda = \omega_{1/2}$

If $F\left[(k+1)\,\omega_m/2 + (k-1)\lambda\right] = 0$ for all values of λ except at $\lambda = \omega_1/2$ then we must have $k=1$ and hence $\omega_m = \omega_i$

If $F^*\left[(k-1)\,\omega_m/2 + (k+1)\lambda\right] = 0$ for all values of λ except at $\lambda = \omega_1/2$ then we must have $k = -1$ and hence $\omega_m = -\omega_i$

Thus for $G(\omega)$ to be rational, only two values of k, namely $k=\pm 1$, are allowed whereas if no condition is imposed on $G(\omega)$, $k=\pm 1$ are two of the possible allowed values of k

3 4 GENERAL LINEAR TRANSFORMATION OF THE t-ω PLANE

In this section we consider a general linear transformation of the t-ω plane.

Let $W_{f,f}(t,\omega) \longleftrightarrow f(t)$

$W_{f_k,f_k}(t,\omega) \longleftrightarrow f_k(t)$

where f(t) is a time signal and $f_k(t)$ is a signal such that

$$W_{f_k,f_k}(t,\omega) = W_{f,f}(t, k\omega) \qquad . (3\ 81)$$

where k is a real constant

$$\text{Let } g(t) = \sqrt{|a|}\ f_k(at)\ e^{jbt^2} \qquad \text{--}(3\ 82)$$

where a and b are real constants

$$\text{Then } W_{g,g}(t,\omega) - W_{f_k,f_k}(at, \frac{k}{a}(\omega - 2bt)) \qquad . (3\ 83)$$

by using equations (3 52) and (3 53)

$$\text{Let } h(t) = F\,T^{-1}\left[G(\omega)e^{jd\omega^2}\right]$$

where d is a real constant

$$W_{h,h}(t,\omega) = W_{f,f}\left[a(t+2d\omega), \frac{k}{a}(\omega - 2b(t+2d\omega))\right] \qquad . (3\ 85)$$

by using equation (3 54)

This means that if we have a linear transformation of the t-ω plane of the type

$$\begin{bmatrix} t' \\ \omega' \end{bmatrix} = \begin{bmatrix} a & 2ad \\ \frac{-2bk}{a} & \frac{(1-4db)k}{a} \end{bmatrix} \begin{bmatrix} t \\ \omega \end{bmatrix} \tag{3 86}$$

then $W_{f,f}(t',\omega') = W_{h,h}(t,\omega)$

But for the existence of $W_{f_k,f_k}(t,\omega)$ and hence in turn for the existence of $W_{h,h}(t,\omega)$, we need k such that

$$F[(k+1)\omega + (k-1)\lambda] \quad F^*[(k-1)\omega + (k+1)\lambda]$$

$$= \frac{F[(k-1)\lambda] \cdot F^*[(k+1)\lambda] \cdot F[(k+1)\omega] \quad F^*[(k-1)\omega]}{|F(0)|^2} \tag{3 87}$$

as given by equation (3 76) with $x = \omega$ and $y = \lambda$

From the above discussion it follows that if we consider a general linear transformation of the $t-\omega$ plane

$$\begin{bmatrix} t' \\ \omega' \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{22} & l_{22} \end{bmatrix} \begin{bmatrix} t \\ \omega \end{bmatrix} \tag{3 88}$$

then by comparison we get

$$a = l_{11} \quad , \quad d = \frac{1}{2} \quad \frac{l_{12}}{l_{11}}$$
$$b = \frac{-l_{11} l_{21}}{2(\text{det. } D)} \quad , \quad k = (\text{det. } D) \qquad (3\ 89)$$

where (det. D) denotes the determinant of the transformation matrix D

The transformation will be invariant to the WD if(det D)-k satisfies the condition mentioned in equation (3 87) However since $k = \pm 1$ are valid values of k we can claim that any matrix of transformation D is valid as long as

$$\det D = \pm 1 \qquad (3\ 90)$$

3 5 NONLINEAR TRANSFORMATION OF THE t-ω PLANE

In this section we will consider any general transformation of the t-ω plane For the sake of simplicity of analysis we will assume that only ω-axis has been tranformed. The result for t-axis transformation follows exactly the same pattern and hence will not be discussed here

Let $\omega' = u(\omega)$ be a general transformation of the ω-axis

Let $W_{f,f}(t,\omega) \longleftrightarrow f(t)$

and $C(t,\omega) = W_{f,f}(t, u(\omega)) \qquad (3\ 91)$

Let us find the condition under which the transformation $u(\omega)$ is invariant to the WD, i e we wish to find condition under which $C(t, \omega)$ is also an auto WD of some signal

$$\text{Let} \quad T_c(x,y) = \int_{-\infty}^{+\infty} C(t, \tfrac{x-y}{2})\, e^{-jt(x-y)} dt \qquad (3\ 92)$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t+\tfrac{\eta}{2})\, f^*(t-\tfrac{\eta}{2})\, e^{-j\eta u(\frac{x+y}{2})}\, e^{-jt(x-y)}\, d\eta\, dt$$

Let $t+\eta/2 = p$ $\qquad t-\eta/2 = q$

$$T_c(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(p) f^*(q)\, e^{-ju(\frac{x+y}{2})(p-q)}\, e^{-j(x-y)(\frac{p+q}{2})}\, dp\,dq$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(p)\, e^{-jp\left[(\frac{x-y}{2}) + u(\frac{x+y}{2})\right]}\, f^*(q)\, e^{jq\left[-\frac{x-y}{2} + u(\frac{x+y}{2})\right]}\, dp\,dq$$

$$= F\left[u\left(\tfrac{x+y}{2}\right) + \tfrac{x-y}{2}\right]\ F^*\left[u\left(\tfrac{x+y}{2}\right) - \tfrac{x-y}{2}\right]$$

By equation (3 70) $C(t, \omega)$ will be an auto WD and hence the transformation $u(\omega)$ will be invariant to the WD iff

$$F\left[u\left(\frac{x+y}{2}\right) + \frac{x-y}{2}\right] F^{*}\left[u\left(\frac{x+y}{2}\right) - \frac{x-y}{2}\right] \tag{3 93}$$

$$= G(x)\; G^{*}(y)$$

for all x and y and for some spectrum $G(\omega)$

Special Case Let $F(\omega)$ be rational as given by equation (3 77) which is the most commonly found case in practice From the practical point of view we will be interested in having $G(\omega)$ also a rational spectrum Let us find conditions under which $G(\omega)$ remains rational

Let us for a moment assume that $C(\omega)$ is also rational

$$G(\omega) = \frac{\prod_m \omega - \omega_m}{\prod_1 \omega - \omega_1} \tag{3 94}$$

- $T_C(\omega, \lambda) = 0$ for $\omega = \omega_m$, $\forall \lambda$ except $\lambda = \omega_1$.

$$F\left[u\left(\frac{\omega_m + \lambda}{2}\right) + \frac{\omega_m - \lambda}{2}\right] = 0 \tag{3 95}$$

$$\text{or}\quad F^{*}\left[u\left(\frac{\omega_m + \lambda}{2}\right) - \frac{\omega_m - \lambda}{2}\right] = 0 \tag{3 96}$$

$\forall \lambda$ except $\lambda = \omega_1$

Equation (3 95) implies

$$u\left(\frac{\omega_m + \lambda}{2}\right) + \frac{\omega_m - \lambda}{2} = \omega_i$$

Let $\frac{\omega_m + \lambda}{2} = \Psi$

$$u(\Psi) = \omega_i - \frac{\omega_m}{2} + \Psi - \frac{\omega_m}{2}$$

$$= (\omega_i - \omega_m) + \Psi$$

$$= K_{im} + \Psi \qquad (3\ 97)$$

where K_{im} is a constant given by

$$K_{im} = \omega_i - \omega_m$$

Thus for all possible values of K_{im}, we get a transformation which transforms the ω -axis as given by equation (3 97)

Similarly equation (3 96) implies

$$f(\Psi) = K_{ij} - \Psi \qquad (3\ 98)$$

and we get another set of valid transformations

Hence if we wish to have rational $G(\omega)$ a large number of allowed transformations exist given by equations (3.97) and

(3 98) although the nature of these transformations is the same for all It is important to note that all these transformations are linear implying that no nonlinear transformation of the ω-axis exist if we wish that a rational spectrum $F(\omega)$ should give rise to another rational spectrum $G(\omega)$ after the transformation

3 6 REALIZABILITY CONDITION OF THE WIGNER DISTRIBUTION BASED ON THE BASIS-VECTORS

For any given time-frequency distribution $C(t,\omega)$ if it is possible to find a signal $f(t)$ such that

$$f(t) \longleftrightarrow W_{f,f}(t,\omega) = C(t,\omega) \tag{3 99}$$

then we say that $C(t,\omega)$ is WD realizable Realizability condition for the WD has been stated in property 24 of section 3 3 and can be derived with the help of basis-vectors in the same fashion as in the case of Ambiguity Function [34] .

Let $\{\varphi_m(t)\}$ be a complete orthonormal set of basis functions Hence a time signal $f(t)$ can be expressed as

$$f(t) = \sum_{i=-\infty}^{+\infty} a_i \varphi_i(t) \tag{3 100}$$

where the coefficients a_is are given by the equation

$$a_i = \int_{-\infty}^{+\infty} f(t)\ \varphi_i(t)\ dt \tag{3 101}$$

$$W_{f,f}(t,\omega) = \int_{-\infty}^{+\infty} \sum_{i=-\infty}^{+\infty} a_i \varphi_i(\tau+\tfrac{\eta}{2}) \sum_{j=-\infty}^{+\infty} a_j \varphi_j^*(t-\tfrac{\eta}{2}) e^{-j\omega\eta}\ d\eta.$$

$$= \sum_i \sum_j a_i a_j^* \int_{-\infty}^{+\infty} \varphi_i(t+\tfrac{\eta}{2}) \varphi_j^*(t-\tfrac{\eta}{2})\ e^{-j\omega\eta}\ d\eta \tag{3 102}$$

Let $$D_{i,j}(t,\omega) = \int_{-\infty}^{+\infty} \varphi_i(t+\tfrac{\eta}{2})\ \varphi_j^*(t-\tfrac{\eta}{2})\ e^{-j\omega\eta}\ d\eta \tag{3 103}$$

$$\bullet\ W_{f,f}(t,\omega) = \sum_i \sum_j a_i\ a_j^*\ D_{i,j} \tag{3 104}$$

It can be shown that $\left\{D_{i,j}(t,\omega)\right\}$ forms a complete orthonromal set of basis functions of variables t and ω This statement can be proved with the help of a previous result namely that $\left\{A_{i,j}(\tau,\sigma)\right\}$ forms a complete orthonormal set of basis functions[34] , where $A_{i,j}(\tau,\sigma)$ is the Ambiguity Function of the basis functions $\varphi_i(t)$ and $\varphi_j(t)$ defined in equation (3.43)

Using equation (3 46) we get

$$D_{i,j}(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{i,j}(\tau,\sigma)\, e^{-j(\tau\omega-\sigma t)}\, d\tau\, d\sigma$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D_{i,j}(t,\omega)\ D_{m,n}(t,\omega)\ dt\ d\omega$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{(2\pi)^3} A_{i,j}(\tau,\sigma)\ A_{m,n}(\alpha,\beta)\ e^{-j(\tau\omega-\sigma t)}$$

$$e^{+j(\alpha\omega-\beta t)} d\alpha\, d\beta\, d\tau\, d\sigma\ dt\, d\omega$$

$$= \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{i,j}(\tau,\sigma) A^{*}_{m,n}(\alpha,\beta)\ e^{jt(\sigma-\beta)}$$

$$e^{-j\omega(\tau-\alpha)}\ d\alpha\, d\beta\, d\tau\, d\sigma\ dt\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{i,j}(\tau,\sigma)\ A^{*}_{m,n}(\alpha,\beta)\,\delta(\sigma-\beta)\,\delta(\tau-\alpha)\, d\alpha\, d\beta\, d\tau\, d\sigma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} A_{i,j}(\tau,\sigma)\ A^{*}_{m,n}(\tau,\sigma)\ d\tau\, d\sigma \qquad (3\ \ 105)$$

Hence $\left\{D_{i,j}(t,\omega)\right\}$ forms a complete orthonormal set of basis

functions of variables t and ω This implies that the time-frequency distribution $C(t, \omega)$ can be written as

$$C(t, \omega) \quad \sum_i\sum_j B_{ij} \, D_{i,j} \tag{3.106}$$

where the coefficient B_{ij} is given by

$$B_{ij} = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} C(t,\omega) D^*_{i,j}(t,\omega) \, dt \, d\omega \tag{3 107}$$

Comparing this with equation (3 104), we find that $C(t,\omega)$ can be an auto WD iff

$$B_{ij} = a_i \, a_j^* \quad \text{for all } i, j$$

i e iff $\underline{\underline{B}} = \underline{a} \, \underline{a}^{*T}$ (3.108)

where $\underline{\underline{B}}$ is a square matrix with $(i,j)^{th}$ element as B_{ij} and $\underline{a}$ is a column matrix with i^{th} element as a_i Similarly it can be stated that $C(t, \omega)$ can be a cross WD of same time signals f(t) and g(t) iff

$$\underline{\underline{B}} = \underline{a} \, \underline{b}^{*T} \tag{3 109}$$

where elements of the matrix $\underline{b}$ are given by an equation similar to equation (3.101)

$$b_i = \int_{-\infty}^{+\infty} g(t) \, \varphi_i(t) \, dt \quad . \tag{3 110}$$

Factorability of the square matrix $\underline{\underline{B}}$ into the simple or outer product of two column matrices $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as given by equation (3 109) implies and is implied by unity rank of $\underline{\underline{B}}$ [35] Moreover by equation (3 107)

$$B^{*}_{ji} = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} C^{*}(t, \omega)\, D_{j,i}(t,\omega)\, dt d\omega$$

$$= \frac{1}{2\pi} \iint_{-\infty}^{+\infty} C^{*}(t, \omega)\, D^{*}_{i,j}(t,\omega)\, dt d\omega \qquad (3\ 111)$$

Using equation (3 16)

$$B_{ij} = B^{*}_{ji} \quad \text{iff } C(t, \omega) = C^{*}(t, \omega)$$

But $C(t, \omega) = C^{*}(t, \omega)$ only if it is an auto WD. Hence we get the following result $C(t,\omega)$ is a WD iff rank of $\underline{\underline{B}}$ is unity and moreover if

$$\underline{\underline{B}} = \underline{\underline{B}}^{*T} \qquad (3.112)$$

then $C(t,\omega)$ is an auto WD

3 7 REALIZABILITY CONDITION FOR LINEAR INTEGRAL TRANSFORMATION

A linear system in general has following input-output relationship

$$g(t) = \int_{-\infty}^{+\infty} K(t,\tau)\; f(\tau)\; d\tau \qquad (3\ 113)$$

where $f(t)$ is the input, $g(t)$ is the output and $K(t,\tau)$ is the impulse response It can be shown by substitution that the WD of the output is given by

$$W_{gg}(t,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W_{k,k}(t,\omega,\tau,\sigma)\; W_{f,f}(\tau,\sigma)\; d\tau d\sigma \qquad (3\ 114)$$

where $W_{k,k}(t,\omega,\tau,\sigma)$ is a two-dimensional WD of the impulse response $K(t,\tau)$ defined as

$$W_{k,k}(t,\omega,\tau,\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(t+\frac{\eta}{2},\tau+\frac{\eta'}{2})K^*(t-\frac{\eta}{2},\tau-\frac{\eta'}{2})$$

$$e^{-j(\omega\eta-\sigma\eta')}\; d\eta d\eta' \qquad (3.115)$$

In general suppose a kernel $X(t,\omega,\tau,\sigma)$ is given to us and we wish to ascertain the invariance of this kernel to the WD. The invariance condition will be satisfied if $X(t,\omega,\tau,\sigma)$ corresponds to a two-dimensional WD of some linear system impulse response This in turn implies and is implied by the fact that the linear transformation of the WD of any signal using this kernel is also a WD Hence we wish to find whether

$$B(t,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lambda(t,\omega,\tau,\sigma)\, W_{ff}(\tau,\sigma)\, d\tau d\sigma \tag{3 116}$$

correspond to an auto WD of some signal or not

Now according to equation (3.68), $B(t,\omega)$ can be an auto WD iff

$$T_B(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} B(\tfrac{x+y}{2},\omega)\, e^{j\omega(x-y)}\, d\omega = g(x)g^*(y) \quad . \tag{3 117}$$

for some signal $g(t)$

Now L H S of the above equation can be written as

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \chi(\tfrac{x+y}{2},\omega,\tau,\sigma)\, W_{f,f}(\tau,\sigma)\, e^{j\omega(x-y)} d\omega d\tau d\sigma \tag{3 118}$$

Let $$\int_{-\infty}^{+\infty} \chi(\tfrac{x+y}{2},\omega,\tau,\sigma)\, e^{j\omega(x-y)}\, d\omega = \lambda(x,y,\tau,\sigma) \tag{3 119}$$

$$T_B(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \lambda(x,y,\tau,\sigma)\, W_{f,f}(\tau,\sigma)\, d\tau d\sigma \tag{3.120}$$

Let us expand $W_{f,f}(\tau,\sigma)$ and $\lambda(x,y,\tau,\sigma)$ using the complete orthonormal set $\{D_{ij}(t,\omega)\}$ discussed in section 3 6 and defined by equation (3 103)

$$W_{f,f}(\tau,\sigma) = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} a_i a_j^* D_{i,j}(\tau,\sigma) \quad (3.121)$$

and $$\lambda(x,y,\tau,\sigma) = \sum_{i=-\infty}^{+\infty} \sum_{j=-\infty}^{+\infty} C_{i,j}(x,y) D_{i,j}(\tau,\sigma) \quad (3.122)$$

where the coefficients a_i s are defined in equation (3.101) and coefficients $C_{i,j}(x,y)$ are defined as

$$C_{i,j}(x,y) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \lambda(x,y,\tau,\sigma) D_{i,j}(\tau,\sigma)\, d\tau d\sigma \quad (3.123)$$

$$T_B(x,y) = \sum_i \sum_j a_i a_j^* C_{i,j}(x,y) \quad (3.124)$$

Thus $B(t,\omega)$ can be an auto WD iff

$$\sum_i \sum_j a_i a_j^* C_{i,j}(x,y) = v(x)v^*(y) \quad (3.125)$$

for some function $v(x)$.

A sufficient condition for this to happen is

$$C_{i,j}(x,y) = S_i(x) S_j^*(x) \quad (3.126)$$

for some function $S_n(x)$. But the converse is not proved so far

Special Case

Let $X(t,\omega,\tau,\sigma) = X(t-\tau,\omega-\sigma)$ (3 127)

which means if $X(t,\omega,\tau,\sigma)$ happens to be a two-dimensional WD of the impulse response of a linear-system then that linear system is time-invariant Hence equation (3 116) becomes

$$B(t,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(t-\tau,\omega-\sigma)\, W_{f,f}(\tau,\sigma)\, d\tau d\sigma \qquad (3\ 128)$$

Let $t-\tau = \eta, \quad \omega-\sigma = \Psi$

$$B(t,\omega) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(\eta,\Psi)\, W_{f,f}(t-\eta,\omega-\Psi)\, d\eta d\Psi$$

$$\cdot\ T_B(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} X(\eta,\Psi)\, W_{f,f}\left(\frac{x+y}{2}-\eta,\ \omega-\Psi\right) e^{j\omega(x-y)}\, d\omega d\eta d\Psi$$

Now

$$\int_{-\infty}^{+\infty} W_{f,f}\left(\frac{x+y}{2}-\eta,\ \omega-\Psi\right) e^{j\omega(x-y)}\, d\omega$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} F\left(\omega-\Psi+\frac{\varphi}{2}\right) F^*\left(\omega-\Psi-\frac{\varphi}{2}\right) e^{j\varphi\left(\frac{x+y}{2}-\eta\right)} e^{j\omega(x-y)}\, d\varphi d\omega$$

Let $\omega - \Psi + \frac{\omega}{2} = p \qquad \omega - \Psi - \frac{\varphi}{2} = q$

Above integration becomes

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(p)\, F^*(q)\, e^{j\left(\frac{x+y}{2} - \eta\right)(p-q)}\, e^{j(x-y)\left(\frac{p-q}{2} + \Psi\right)}\, dp\, dq$$

$$= e^{j(x-y)\Psi} \iint_{-\infty}^{+\infty} F(p)\, e^{jp\left(\frac{x+y}{2} - \frac{x-y}{2} - \eta\right)}\, F^*(q)\, e^{-jq\left(\frac{x+y}{2} - \frac{x-y}{2} - \eta\right)}\, dp\, dq$$

$$= e^{j(x-y)\Psi} f(x-\eta)\, f^*(y-\eta)$$

$$T_B(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(\eta, \Psi)\, e^{j(x-y)}\, f(x-\eta)\, f^*(y-\eta)\, d\eta\, d\Psi$$

. $B(t, \omega)$ is an auto WD corresponding to some signal iff

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(\eta, \Psi)\, e^{j(x-y)\Psi}\, f(x-\eta)\, f^*(y-\eta)\, d\eta\, d\Psi = v(x)\, v^*(y) \qquad . \ (3\ 129)$$

for some function $v(x)$

3.8 DISCRETE WD FORMULATION

Computation of the WD values involves evaluation of a continuous integration which ultimately will be done by using numerical methods This inherently involves some error and a lot of computational efforts To overcome these drawbacks we would like to define discrete WD in a manner analogous to the definition of the continuous WD This new definition will use discrete time signal (which can be obtained by sampling the continuous time signal), and all the integrations will be replaced by summations, simplifying the computations in the process This definition as given by Claasen and Mecklenbrauker [33] for two discrete-time signals f(n) and g(m) is as follows

$$W_{f,g}(n,\omega) = 2 \sum_{k=-\infty}^{+\infty} f(n+k)g^{*}(n-k)e^{-j2k\omega} \qquad (3\ 130)$$

where ω is a continuous frequency variable To simplify the computations further even the frequency variable should be discretized But this should be done in such a way that all the results and properties of the continuous WD should remain intact In the following analysis the criterion used for sampling the continuous frequency variable ω is to have sampled basis vector $\left\{D_{i,j}(n,m)\right\}$ analogous to the continuous

basis vector $\left\{D_{m,n}(t,\omega)\right\}$ defined in (3 103) This treatment is also an extention of a similar treatment on the Ambiguity Function [34]

Let $f(k)$ be a discrete time signal defined for

$k = -K, -(K-1), ---- 0, --, (K-1), K$ and

$f(k) = 0 \quad \text{for } |k| > K$

Let $\left\{\varphi_i(k)\right\}$ be sampled orthonormal basis set over the interval $-K$ to $+K$, which implies that

$$\varphi_i(k) = 0 \quad \text{for } |k| > K \qquad (3\ 131)$$

and
$$\sum_{k=-K}^{+K} \varphi_i(k)\, \varphi_j^*(k) = \delta_{ij} \qquad (3\ 132)$$

Therefore any discrete time-signal $f(k)$ can be expressed as

$$f(k) = \sum_{i=-K}^{+K} a_i\, \varphi_i(k) \qquad \text{for } |k| < K \qquad (3\ 133)$$

To obtain sampled basis vector $D_{i,j}(n,m)$ we start with the corresponding definition in analogue case given by equation (3 103)

$$D_{i,j}(t,\omega) = \int_{-\infty}^{+\infty} \varphi_i(t+\tfrac{\eta}{2})\, \varphi_j^*(t-\tfrac{\eta}{2})\, e^{-j\omega\eta}\, d\eta$$

Let $t + \frac{\eta}{2} = \beta$

$$D_{ij}(t, \omega) = 2\, e^{j2\omega t} \int_{-\infty}^{+\infty} \varphi_i(\beta)\, \varphi_j(2t-\beta) e^{-j2\omega\beta} d\beta \qquad (3\ 134)$$

Observing this equation let us define the sampled basis vector $D_{i,j}(n,m)$ as follows

$$D_{i,j}(n,m) = 2e^{-j\,2nm2\pi/P} \sum_{k=-K}^{K} \varphi_i(k)\varphi_j^*(2n-k)\; e^{-j2\ 2\pi mk/P} \qquad (3\ 135)$$

which is obtained after replacing t and of equation (3 134) by n and $\frac{2\pi m}{P}$ respectively, where P is an integor This means that sample points in the frequency domain are distanced $\frac{2\pi}{P}$ radians

Let n and m take values from -N to N and -M to M respectively where N and M are some integers

Let us define discrete inner product of $D_{ij}(n,m)$ and $D_{p,q}(n,m)$ as

$$\Delta_{i,j,p,q} = \sum_{n=-N}^{N} \sum_{m=-M}^{M} D_{i,j}(n,m)\; D^*_{p,q}(n,m) \quad . \ (3\ 136)$$

Using equation (3 135) and rearranging we get

$$\Delta_{i,j,p,q} = \sum_{n=-N}^{N} \sum_{k=-K}^{K} \sum_{h=-K}^{K} \varphi_i(k)\varphi_j^*(2n-k)\varphi_p^*(h)\varphi_q(n-h)$$

$$\sum_{m=-M}^{M} e^{-j(k-h)4\pi m/P}$$

The summation under m can be shown to be equal to [34]

$$\frac{\sin\{(M+\frac{1}{2})\ (k-h)\ 4\pi/P\}}{\sin\{(k-h)\ 2\pi/P\}}$$

Let $M = K$ and $P = 2\ (2K+1)$

$$\text{Summation under m} = \frac{\sin \pi(k-h)}{\sin \frac{\pi(k-h)}{(2K+1)}} = (2K+1)\,\delta_{k,h}$$

for $|k|, |h| < K$

$$\Delta_{i,j,p,q} = (2K+1) \sum_{k=-K}^{+K} \sum_{n=-N}^{N} \varphi_i(k)\,\varphi_p^*(k)\,\varphi_j^*(2n-k)\,\varphi_q(2n-k)$$

$$= (2K+1) \sum_{k=-K}^{K} \sum_{n'=-2N-k}^{2N-k} \varphi_i(k)\,\varphi_p^*(k)\,\varphi_j^*(n')\,\varphi_q(n')$$

where $n' = 2n-k$

Now if we put $N=K$ we find that for any value of k between $-K$ to K, n' always covers the range $-K$ to $+K$

Hence $$\Delta_{i,j,p,q} = (2K+1)\,\delta_{ip}\,\delta_{jq} \tag{3 137}$$

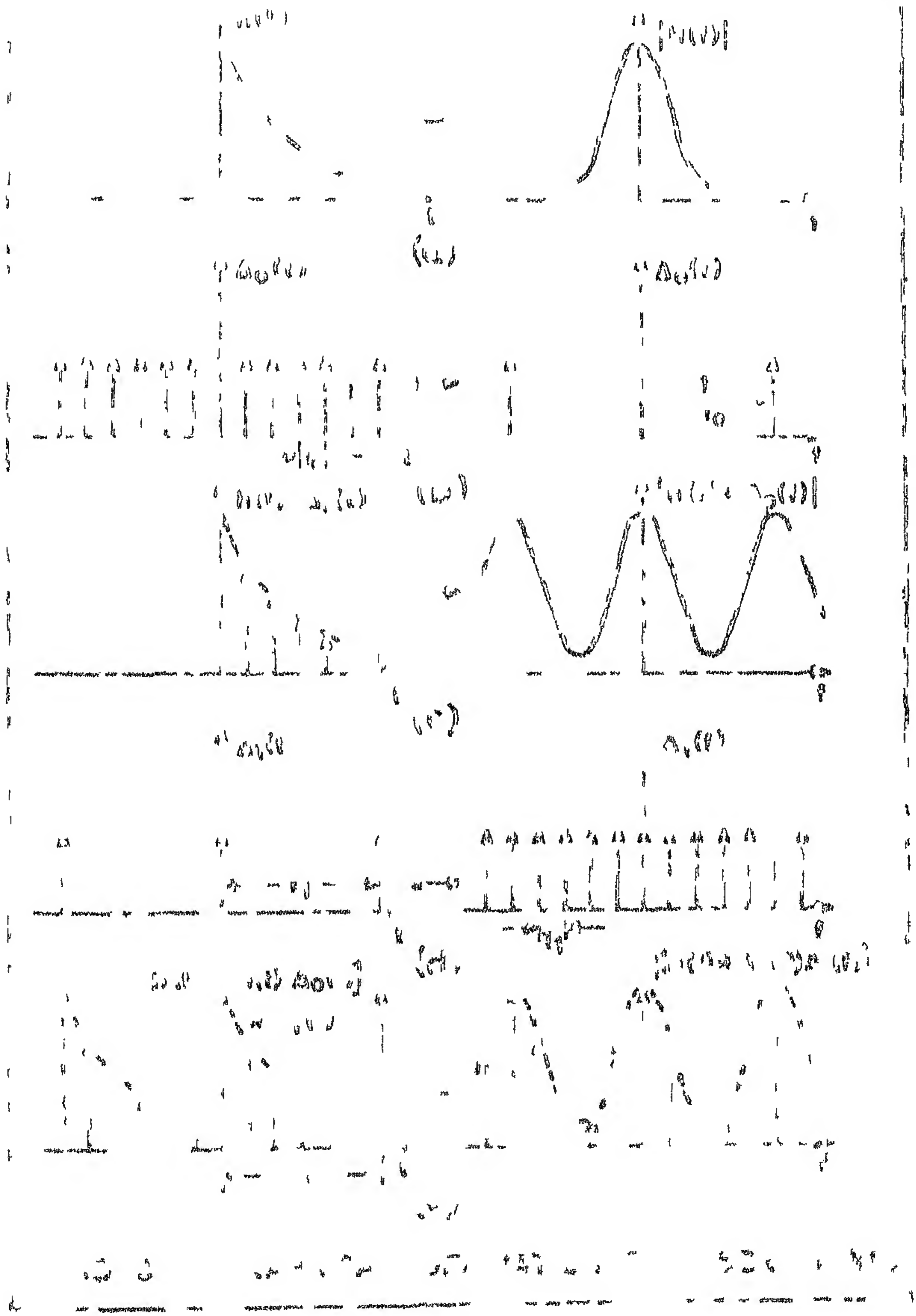
Thus the basis vectors $\{D_{i,j}(n,m)\}$ are orthogonal to each other, a result which we were aiming at

. with $N = K$, $M = K$, $P = 2(2K+1)$ we have $\{D_{i,j}(n,m)\}$ as complete orthonormal set of basis vectors for the discrete WD This means we should have $(2K+1)$ points in time domain spaced equally and $(2K+1)$ points in frequency domain spaced at a distance of $\frac{\pi}{(2K+1)}$ radians With this framework the results derived in sections 3 2 to 3 7 follow for discrete WD also

3 9 FAST COMPUTATION OF THE WD

As has been mentioned in the last section for the purpose of actual computation of the WD on a digital computer we have to sample both time and frequency domains For the purpose of fast computations an expression of the WD similar to the Discrete Fourier Transform expression can be derived starting from the analogue signal The Fast Fourier Transform algorithms can then be used directly

Consider an analogue signal $h(t)$ and its Fourier Transform $H(f)$ as shown in Fig 3 1(a) To sample the signal in time domain $h(t)$ is multiplied by a pulse train $\Delta_0(t)$ and to sample it in frequency domain the Fourier Transform of the resulting signal is multiplied by a frequency domain pulse train $\Delta_1(f)$ The time periods of $\Delta_0(t)$ and $\Delta_1(t)$ are T_0 and T_1



respectively. In order to have minimum aliasing effect we must have $T_1 = NT_0$, where N is the number of time domain samples which approximate the signal h(t) (i e h(t) is negligible after N samples) Fig 3 1(b) to (d) show the sampling operation graphically The final signal $\tilde{h}(t)$ can be written as

$$\tilde{h}(t) = T_1 \sum_{r=-\infty}^{+\infty} \sum_{k=0}^{N-1} h(kT_0)\ \delta(t-kT_0-rT_1) \qquad (3\ 138)$$

It can be argued that the W D of $\tilde{h}(t)$ is nonzero only for

$$t = \frac{k}{2} T_0 \qquad k=0, 1, \pm 2, ---$$

$$\text{and } \omega = \pi l f_1 \qquad l=0, \pm 1, \pm 2, ---$$

$$\text{where} \qquad f_1 = \frac{1}{T_1}$$

Also the WD is periodic in both time and frequency domains with periods T_1 and $f_0 (=\frac{1}{T_0})$ respectively Let us try to compute the WD value at points

$$t = k\ T_0 \qquad k = 0, 1, 2, ---, (N-1)$$

$$\omega = 2\pi n f_1 \qquad n = 0, 1, 2, ---, (N-1)$$

By looking at the graphical picture and applying the definition of the WD we get,

$$W_{\tilde{h}\tilde{h}}(t-kT_0, \ \omega -2\pi nf_1)$$

$$= \sum_{P=-\infty}^{+\infty} \tilde{h}(kT_0+pT_0)\ \tilde{h}^* (kT_0-pT_0) e^{-j(2\pi nf_1)2PT_0}$$

$$= \sum_{P=-\infty}^{+\infty} \tilde{h}(kT_0+pT_0)\tilde{h}^*(kT_0-pT_0) e^{-j\,4\pi np/N} \qquad (3\ 139)$$

Since $\tilde{h}(t)$ is periodic we have

$$\tilde{h}(kT_0) = \tilde{h}(kT_0 \pm rT_1) \qquad r = 0, \pm 1, \pm 2, \ldots$$

Equation (3 139) can be written as

$$W_{\tilde{h}\tilde{h}}(t = kT_0, \ \omega = 2\pi nf_1)$$

$$= \sum_{r=-\infty}^{+\infty} \sum_{l=0}^{N-1} \tilde{h}(kT_0+lT_0)\ \tilde{h}^* (kT_0-lT_0) e^{-j4\pi n(l+rN)/N}$$

$$= \sum_{r=-\infty}^{+\infty} \sum_{l=0}^{N-1} \tilde{h}(kT_0+lT_0)\tilde{h}^*(kT_0-lT_0) e^{\frac{-j4\pi nl}{N}} \ e^{-j4\pi nr}$$

$$= \sum_{l=0}^{N-1} \tilde{h}(kT_0+lT_0)\ \tilde{h}^*(kT_0-lT_0) e^{\frac{-j4\pi nl}{N}} \ \sum_{r=-\infty}^{\infty} e^{-j4\pi nr}$$

The summation under r contributes a constant independent of k and n and hence can be neglected Also

$$\tilde{h}(mT_0) = h(m'T_0) \tag{3 140}$$

where $m = 0, \pm 1, \pm 2, \ldots$

and $m' = m \pmod N$

We get

$$W_{\tilde{h}h}(t = kT_0, \quad \omega = 2\pi n f_1)$$

$$\sum_{l=0}^{N-1} h \quad (k \oplus l)\, T_0 \quad h^* \quad (k \ominus l)\, T_0 \quad e^{-j4\pi nl/N} \tag{3 141}$$

where $\oplus$ indicate mod-N addition and $\ominus$ indicate mod-N subtraction For the sake of simplicity in writing we can omit explicit mention of T_0 and f_1 and write

$$W_{h\tilde{h}}(k,n) = \sum_{l=0}^{N-1} h(k \oplus l)\, h^*(k \ominus l) e^{-j4\pi nl/N}$$

Let $h(k \oplus l)^* h(k \ominus l) = g(k,l)$ (3 142)

$$W_{\tilde{h}\tilde{h}}(k,n) = \sum_{l=0}^{N-1} g(k,l) e^{-j4\pi nl/N} \tag{3 143}$$

To make this expression look like Discrete Fourier Transform expression let $n = m/2$

$$W_{\tilde{h},\tilde{h}}(k,m/2) = \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi ml/N} \qquad (3.144)$$

For a constant k / (3 144) looks exactly like the Discrete Fourier Transform equation Hence to compute the WD values at points

$$t = kT_0, \qquad \omega = \frac{2\pi m f_1}{2}, \; m = 0,1,2, \ldots, (N-1)$$

we can use the Fast Fourier Transform (FFT) algorithms provided we compute the value of the function $g(k,l)$ for $l = 0,1, \ldots, (N-1)$ before hand Thus at time $t = kT_0$ we can get the WD values at frequencies from zero to $\pi(N-1)f_1$ at the interval of πf_1 radians The WD values for further frequencies can be obtained from these WD values by proper extrapolation as can be seen from Fig 3 1(d) The same procedure is to be repeated to obtain the WD values at other time instants Hence by the repeated use of the FFT algorithms we can obtain the WD values at all time and frequency sample points of interest

3.10 EXAMPLES

In this section some examples have been worked out to illustrate the concept of the WD

1) Let $f(t) = 1 \qquad |t| < T$

$\qquad = 0 \qquad |t| > T$

Then

$$W_{f,f}(t,\omega) = \frac{2}{\omega} \sin[\,2\omega\,(T - t)\,] \qquad |t| < T$$

$$= 0 \qquad |t| > T$$

This WD has a sin x/x shape with respect to frequency just like the Fourier Transform of f(t) The width of the main lobe however depends on the value of t Also the WD is negative in certain regions of the t-ω plane.

2) Let $f(t) = e^{j\omega_o t} \qquad |t| < T$

$\qquad = 0 \qquad |t| > T$

Then $W_{f,f}(t,\omega) = \dfrac{2}{w - w_o} \sin[\,2(w - w_o)\,(T - (t))] \qquad |t| < T$

$\qquad = 0 \qquad |t| > T$

3) Let $f(t) = Ae^{j\omega_o t}$ for all t

Then $W_{f,f}(t,\omega) = |A|^2\, 2\pi\delta(\omega - \omega_o)$

This result is quite satisfactory because this shows for a stationary signal the WD is independent of t and is confined to the line $\omega = \omega_o$ as expected

4) Let $\quad f(t) = A_1 e^{j\omega_1 t}$

$$g(t) = A_2 e^{j\omega_2 t}$$

Then $\quad W_{f,g}(t, \omega) = A_1 A_2^{*} \; e^{j(\omega_1 - \omega_2)t} \quad 2\pi\delta\left(\omega - \frac{\omega_1 + \omega_2}{2}\right)$

5) Let $\quad f(t) = A \cos(\omega_o t + \varphi)$

Then $W_{f,f}(t, \omega) = |A|^2 \; \frac{\pi}{2}\left[\delta(\omega - \omega_o) + \delta(\omega + \omega_o)\right.$

$$\left. 2\delta(\omega) \cos\{2(\omega_o t + \varphi)\}\right]$$

Thus apart from stationary contributions at frequency $\pm\omega_o$, the WD also contains a nonstationary term oscillating with frequency $2\omega_o$

6) Let $\quad f(t) = Ae^{j\ t^2}$

Then $\quad W_{f,f}(t, \omega) = 2\pi |A|^2 \; \delta(\omega - 2\alpha t)$

This again is an extremely satisfying result because it clearly shows the concentration of the WD at frequency $\omega = 2\alpha t$ which is the instantaneous frequency of f(t)

7 This example illustrates how the WD changes with a linear transformation of the $t-\omega$ plane For this consider a rectangular pulse of width $2T = 4$ and its WD as mentioned in example 1

$$f(t) = 1 \qquad |t| < 2$$
$$= 0 \qquad |t| > 2$$

and $$W_{f,f}(t, \omega) = \frac{2}{\omega} \sin \{2\omega(2 - t)\} \qquad |t| < 2$$
$$= 0 \qquad |t| > 2$$

Fig 3 1 shows an amplitude contour of the WD for a value of 1 2 which brings into sight only first side lobe Suppose we wish to transform the plane so that the side lobe is off the ω-axis To this end consider following transformation

$$\begin{bmatrix} t' \\ \omega' \end{bmatrix} = \begin{bmatrix} 1 & 8/7 \\ 1/4 & 9/7 \end{bmatrix} \begin{bmatrix} t \\ \omega \end{bmatrix}$$

The determinant of the transformation matrix is 1 as required for the invariance condition The t-axis and ω-axis are transformed to straight lines $9t' = 8\omega'$ and $t' = 4\omega'$, respectively. Fig 3 2 shows the new amplitude contour for the

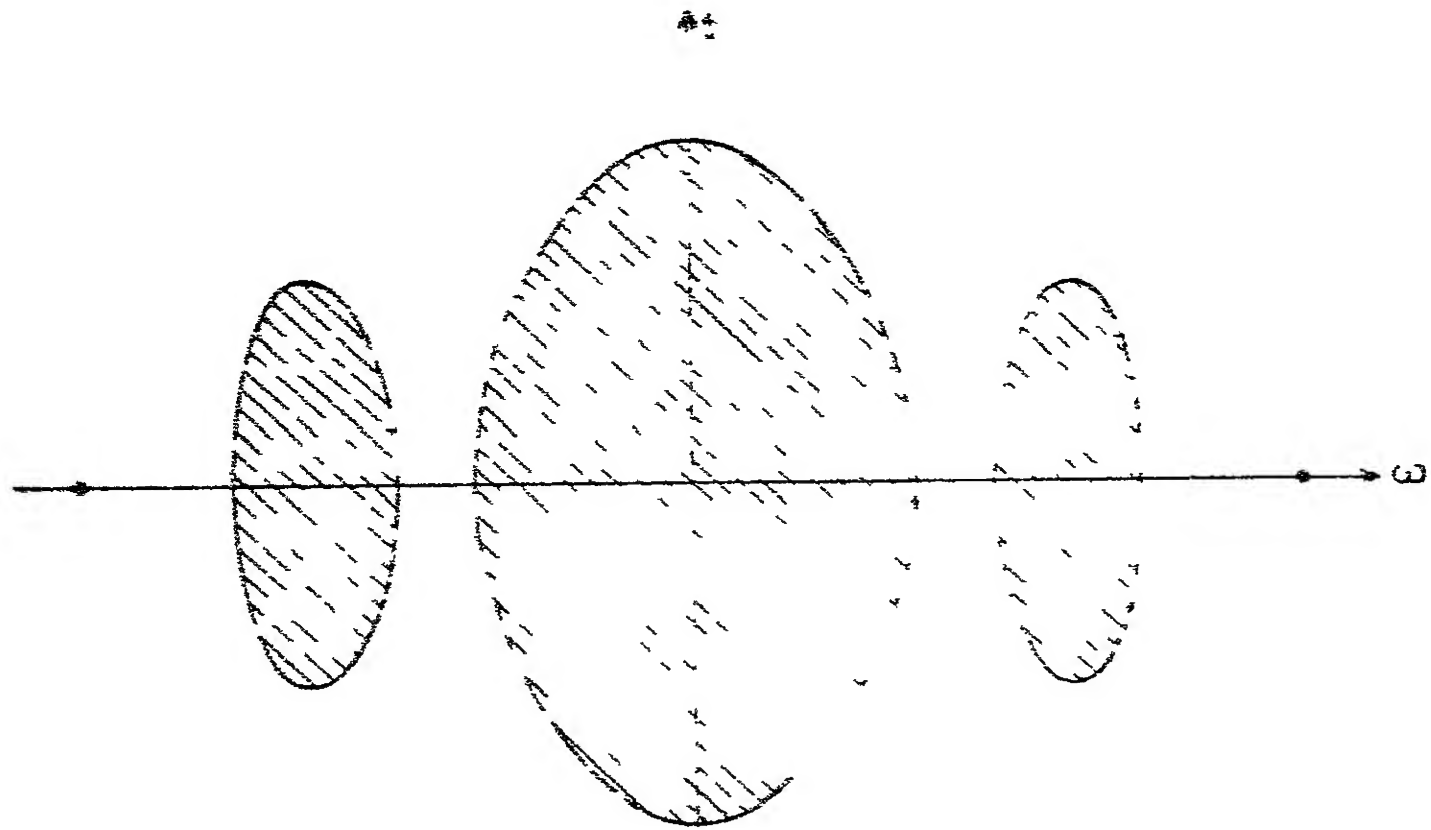
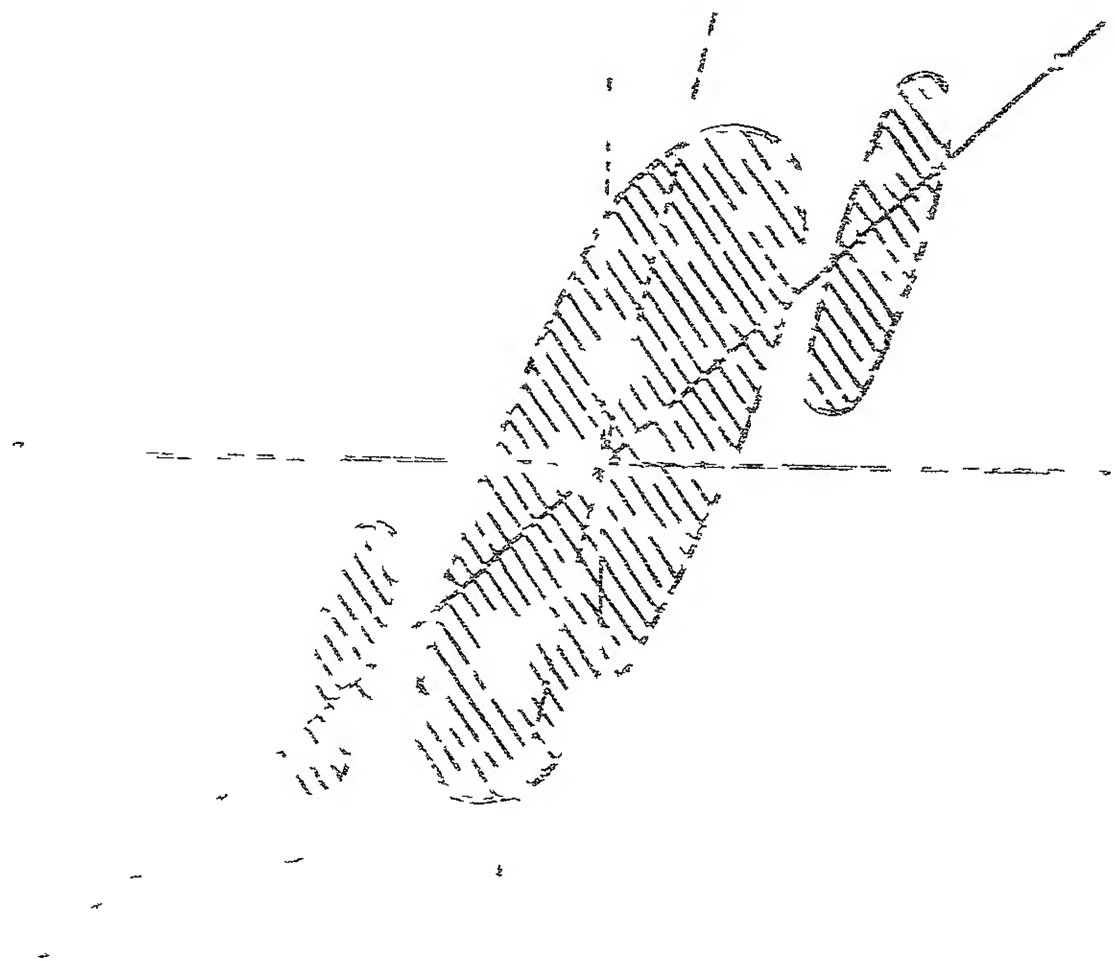


FIG 3 2 AMPLITUDE COUNTOUR OF W.D BEFORE TRANSFORMATION



same amplitude value The new signal $h(t)$, having the transformed WD can be obtained by using the results of section 3 4 and [30] Here

$$a = l_{11} = 1 \qquad d = \frac{l_{12}}{2l_{11}} = \frac{4}{7}$$

$$b = \frac{-l_{11}l_{21}}{2(\det D)} = \frac{-1}{8} \qquad k = \det D = 1$$

$$f_k(t) = f(t)$$

$$g(t) = f(t)\, e^{-jt^2/8}$$

and
$$h(t) = FT^{-1}\{ G(\omega)\, e^{j4\omega^2/7} \}$$

or
$$H(\omega) = e^{j4\omega^2/7} \int_{-\infty}^{+\infty} f(t)\, e^{-jt^2/8}\, e^{-j\omega t}\, dt$$

$$= 2\pi\, e^{-j\frac{18\omega^2}{7}} \left\{ C\left(\frac{2+2\omega}{\sqrt{2\pi}}\right) - C\left(\frac{2-2\omega}{\sqrt{2\pi}}\right) + j\, S\left(\frac{2-2\omega}{\sqrt{2\pi}}\right) - j\, S\left(\frac{2+2\omega}{\sqrt{2\pi}}\right) \right\}$$

where $C(x)$ and $S(x)$ are Fresnel integrals.

CHAPTER 4

APPLICATIONS OF THE WIGNER DISTRIBUTION TO OPTICS

4.1 INTRODUCTION

One of the reasons why the WD has attracted attention of scientists working in the field of signal processing is its successful use in optics. Papoulis introduced the use of Ambiguity Function in optics in 1974 [36]. Because of the similarity in the definitions of the Ambiguity Function and the WD the logical step was to extend the analysis with the help of WD. This work was done by Bastians [3], [7] Bartelt and Brenner [5], [6] Brared, Gram and Schenzie [4] and others. They also introduced many more new applications. This chapter briefly describes some of the application areas to bring out the usefulness of the WD in optics and to give an idea regarding the use of WD in signal processing.

4.2 TERMINOLOGY

Before presenting the applications to optics it is essential to introduce the terminology used in Fourier Optics. An optical signal can be described in the space domain by its

complex amplitude $f(\underline{r})$, where $\underline{r}$ denotes a vecter with space coordinates (x,y) for a two-dimensional signal Equivalently in frequency domain we have the Fourier Transform of the complex amplitude defined as

$$F(\underline{\omega}) = \int f(\underline{r}) \exp\{-j(\underline{\omega}\ \underline{r})\}\, d\underline{r} \qquad (4\ 1)$$

where $\underline{\omega}$ is a two-dimensional frequency vecter, $\int dr$ stands for $\iint_{-\infty}^{+\infty} dxdy$ (for a two-dimensional signal) and $\omega\ r$ indicate the dot-product of vectors ω and $\underline{r}$ Similarly a linear system is defined by its impulse response $g(\underline{r},\underline{\rho})$ or its Fourier Transform $G(\underline{\omega},\underline{\sigma})$ defined as

$$G(\underline{\omega},\underline{\sigma}) = \frac{1}{(4\pi)^2} \iint g(\underline{r},\underline{\rho}) \exp\{-j(\underline{\omega}\ \underline{r} - \underline{\sigma}\ \underline{\rho})\}\, d\underline{r}\, d\underline{\rho} \qquad (4\ 2)$$

The output $f_o(\underline{r})$ of such a linear system with input $f_i(\underline{r})$ is given as usual by

$$f_o(r) = \int g(\underline{r},\underline{\rho})\, f_i(\underline{\rho})\, d\rho \qquad (4\ 3)$$

The WD defined with this terminology reads

$$W_{f,f}(\underline{r},\underline{\omega}) = \int f(\underline{r} + \frac{\underline{r}'}{2})\, f^*(\underline{r} - \frac{\underline{r}'}{2})\, e^{-j\underline{\omega}\cdot\underline{r}'}\, d\underline{r}' \qquad (4\ 4)$$

For a linear system defined by equation (4 3) the WD of the output is given by

$$W_{f_o,f_o}(\underline{r},\underline{\omega}) = W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma})\, W_{f_i,f_i}(\underline{\rho},\underline{\sigma})\, d\underline{\rho}\, d\underline{\sigma} \qquad (4.5)$$

where $W_{f_i,f_i}(\underline{\rho},\underline{\sigma})$ is the WD of the input signal and $W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma})$ is the two-dimensional WD of the impulse response of the system defined as

$$W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma}) = \frac{1}{(4\pi)^2} \iint g\left(\underline{r} + \frac{\underline{r}'}{2}, \underline{\rho} + \frac{\underline{\rho}'}{2}\right)$$

$$\cdot g^*\left(\underline{r} - \frac{\underline{r}'}{2}, \underline{\rho} - \frac{\underline{\rho}'}{2}\right) \exp\{-j(\underline{\omega}\,\underline{r}' - \underline{\sigma}\,\underline{\rho}')\}\, d\underline{r}'\, d\underline{\rho}' \qquad (4.6)$$

The function $W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma})$ can be considered as a response in the space-frequency domain to the input signal

$$W_{f_i,f_i}(\underline{r},\underline{\omega}) = \delta(\underline{r}-\underline{\rho})\ \delta(\underline{\omega}-\underline{\sigma})$$

Of course this signal does not have any physical origin but only mathematical convenience.

In the next section we consider a few elementary linear optical systems. In each case the WD of the impulse response of the system is given along with the WD of the

output signal It will be seen that relating the WDs of the input and output signals is much more convenient than relating the signals directly either in space domain or in frequency domain

4.3 EXAMPLES

1) SPREADLESS SYSTEM LENS

Let the spreadless system be represented by

$$f_o(\underline{r}) = m(\underline{r})\ f_i(\underline{r}) \quad . \tag{4 7}$$

$$W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma}) = \frac{1}{(4\pi)^2}\ W_{m,m}(\underline{r},\underline{\omega}-\underline{\sigma})\ \delta(\underline{r}-\underline{\sigma}) \tag{4 8}$$

and $$W_{f_o,f_o}(\underline{r},\underline{\omega}) = \frac{1}{4\pi^2} \int W_{m,m}(\underline{r},\underline{\omega}-\sigma)\ W_{f_i,f_i}(\underline{r},\underline{\sigma})\ d\underline{\sigma} \quad . \tag{4 9}$$

Equation (4 9) shows convolution for frequency variable $\underline{\omega}$ and multiplication for the space variable $\underline{r}$, a r sult quite convincing for a spreadless system. In the special case of a lens with a focal distance f we have

$$m(\underline{r}) = \exp\{(-j\frac{k}{2f}\ |\underline{r}|^2)\} \tag{4 10}$$

Here the wave number k is defined as

$$k = 2\pi/\lambda \tag{4 11}$$

where λ is the wavelength of light

$$W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma}) = \delta(\underline{r}-\underline{\rho})\ \delta(\underline{\omega}-\frac{k}{f}\underline{r} - \underline{\sigma}) \tag{4 12}$$

and hence

$$W_{f_o,f_o}(\underline{r},\underline{\omega}) = W_{f_i,f_i}(\underline{r},\underline{\omega}-\frac{k}{f}\underline{r}) \quad . \tag{4 13}$$

2) SHIFT-INVARIANT SYSTEM FREE SPACE

Shift-invariant system is the converse of a spreadless system Let

$$F_o(\underline{\omega}) = H(\underline{\omega})\ F_i(\underline{\omega}) \tag{4 14}$$

represent the shift-invariant system

$$\therefore\ W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma}) = W_{h,h}(\underline{r}-\rho,\ \underline{\omega})\cdot\delta(\underline{\omega}-\underline{\sigma}) \tag{4.15}$$

and $$W_{f_o,f_o}(\underline{r},\underline{\omega}) = \int W_{h,h}(\underline{r}-\underline{\rho},\omega)\, W_{f_i,f_i}(\rho,\omega)d\rho \tag{4 16}$$

This shows a convolution for the space variable and a multiplication for the frequency variable a result as per the expectations In the special case of free space over a distance z

we have

$$H(\underline{\omega}) = \exp\{(-j\frac{z}{2k}\left|\underline{\omega}\right|^{2})\} \tag{4 17}$$

where k is defined in equation (4 10)

$$W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma}) = \delta(\underline{r} - \frac{z}{k}\underline{\omega} - \underline{\rho})\,\delta(\underline{\omega}-\underline{\sigma}) \tag{4 18}$$

and hence

$$W_{f_o,f_o}(\underline{r},\underline{\omega}) = W_{f_i,f_i}(\underline{r} - \frac{z}{k}\omega\ ,\ \underline{\omega}) \tag{4 19}$$

3) FOURIER TRANSFORMER

For a fourier transformer whose impulse response reads as

$$g(\underline{r},\underline{\rho}) = \frac{\beta}{2\pi j}\exp\{(-j\beta\underline{r}\ \underline{\rho})\} \tag{4 20}$$

where β is a constant, we have

$$W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma}) = \delta(\beta\underline{r} - \underline{\sigma})\,\delta(\frac{\underline{\omega}}{\beta} + \underline{\rho}) \tag{4 21}$$

and hence

$$W_{f_o,f_o}(\underline{r},\underline{\omega}) = W_{f_i,f_i}(\frac{-\underline{\omega}}{\beta}\ ,\ \beta\underline{r}) \tag{4 22}$$

It can be seen that the space and frequency domains are interchanged as should be for a Fourier Transformer

4) MAGNIFIER

For a magnifier the impulse response is

$$g(\underline{r},\underline{\rho}) = t\,\delta(t\underline{r} - \underline{\rho}) \tag{4 23}$$

where t is a constant

$$\therefore\; W_{g,g}(r,\underline{\omega},\underline{\rho},\underline{\sigma}) = \delta\,(t\underline{r}-\underline{\rho})\,\delta(\frac{\omega}{t} - \underline{\sigma}) \tag{4 24}$$

and hence

$$W_{f_o,f_o}(\underline{r},\underline{\omega}) = W_{f_i,f_i}(\underline{r},\ t\underline{r},\ \frac{\omega}{t}) \quad .. \tag{4 25}$$

5) FRESNEL'S DIFFRACTION

For a Fresnel's diffraction we have

$$g(\underline{r},\underline{\rho}) = \frac{1}{j\lambda z}\exp\,(jkz)\,\exp\,(\frac{jk}{2z}\ \underline{r}-|\underline{\rho}|^{2}) \quad . \tag{4 26}$$

where z is the distance and k is defined in equation (4 11)

$$W_{g,g}(\underline{r},\underline{\omega},\underline{\rho},\underline{\sigma}) = \frac{1}{\lambda^{2}z^{2}}\,\delta\,(\frac{k}{z}(\underline{r}-\underline{\rho}) - \underline{\omega})\,\delta(\frac{k}{z}(\underline{r}-\underline{\rho}) - \underline{\sigma}) \quad . \tag{4 27}$$

and hence

$$W_{f_o,f_o}(\underline{r},\underline{\omega}) = W_{f_i,f_i}(\underline{r} - \frac{z}{k}\,\underline{\omega},\ -\underline{\omega}) \tag{4.28}$$

CHAPTER 5

A RECEPTION SCHEME USING THE WIGENER DISTRIBUTION

5 1 INTRODUCTION

In this chapter a new reception scheme based on the WD is proposed. We consider a transmitter transmitting a series of pulses which reach the receiver after a series of reflections from a target as in the case of radar signals or as in seismic applications Throughout this discussion we assume that the channel is distortionless and the noise is additive white gaussian Thus the received signal r(t) can be written as

$$r(t) = \sum_i a_i f(t-\tau_i) + n(t) \qquad (5\ 1)$$

where a_is are the reflection coefficients, τ_is are the delays involved, f(t) is the transmitted pulse and n(t) is additive white gaussian noise The receiver is required to find out the reflection coefficients a_is and the delays τ_is Since the noise is white gaussian the correlator (or the matched filter) reception is the optimum linear reception scheme as far as the signal to noise ratio (SNR) at the output is concerned

But this reception scheme gives rise to the side-lobe problems and may shield certain return if the reflection coefficient associated with it is small The analysis included in this chapter tries to investigate whether this problem can be overcome by using a reception scheme based on the WD

5 2 MOTIVATION BEHIND USING THE WIGNER DISTRIBUTION IN A RECEPTION SCHEME

Throughout this analysis we assume that the pulse f(t) transmitted by the transmitter is a pulsed chirp (or linear FM) signal with carrier frequency ω_o, i e.

$$f(t) = A\, e^{j\omega_o t}\, e^{j\alpha t^2} \qquad \text{for } 0 \leq t \leq T_o \qquad (5\ 2)$$

$$= 0 \quad \text{otherwise}$$

where T_o is the duration of the pulse and A and α are constants We assume that $\omega_o >> 2\alpha T_o$, which is the approximate bandwidth of the signal f(t)

f(t) can be considered as a product of two signals,

$$g(t) = e^{j\alpha t^2}\, e^{j\omega_o t} \qquad \text{for } -\infty < t < \infty \qquad (5\ 3)$$

and $h(t) = A \quad \text{for } 0 \leq t \leq T_o$

$$= 0 \quad \text{otherwise} \qquad (5\ 4)$$

Let $W_{f,f}(t) \longleftrightarrow f(t)$

$W_{g,g}(t) \longleftrightarrow g(t)$

and $W_{h,h}(t) \longleftrightarrow h(t)$

It can be shown that

$$W_{g,g}(t,\omega) = 2\pi\delta(\omega - \omega_o - 2\alpha t) \quad . \tag{5 5}$$

Using equation (3 61) we get

$$W_{f,f}(t,\omega) = \frac{1}{2\pi} \; W_{g,g}(t,\omega) \overset{\omega}{*} W_{h,h}(t,\omega)$$

$$= W_{h,h}(t, \omega - \omega_o - 2\alpha t) \tag{5 6}$$

Thus the WD of the signal f(t) is same as that of h(t) except for a shift of the ω-axis at each time instant t It can be shown that

$$W_{h,h}(t,\omega) = 4At\,\frac{\sin(2\omega t)}{2\omega t} \qquad \text{for } 0 \le t \le T_o/2$$

$$= 4A(T_o - t)\,\frac{\sin(2\omega(T_o - t))}{2\omega(T_o - t)} \qquad \text{for } \frac{T_o}{2} \le t \le T_o$$

$$= 0 \text{ otherwise} \tag{5 7}$$

A pictorial view of $W_{h,h}(t,\omega)$ at some time instant $t = t_o$ is shown in fig. 5 1(a). The WD of the signal f(t) will be

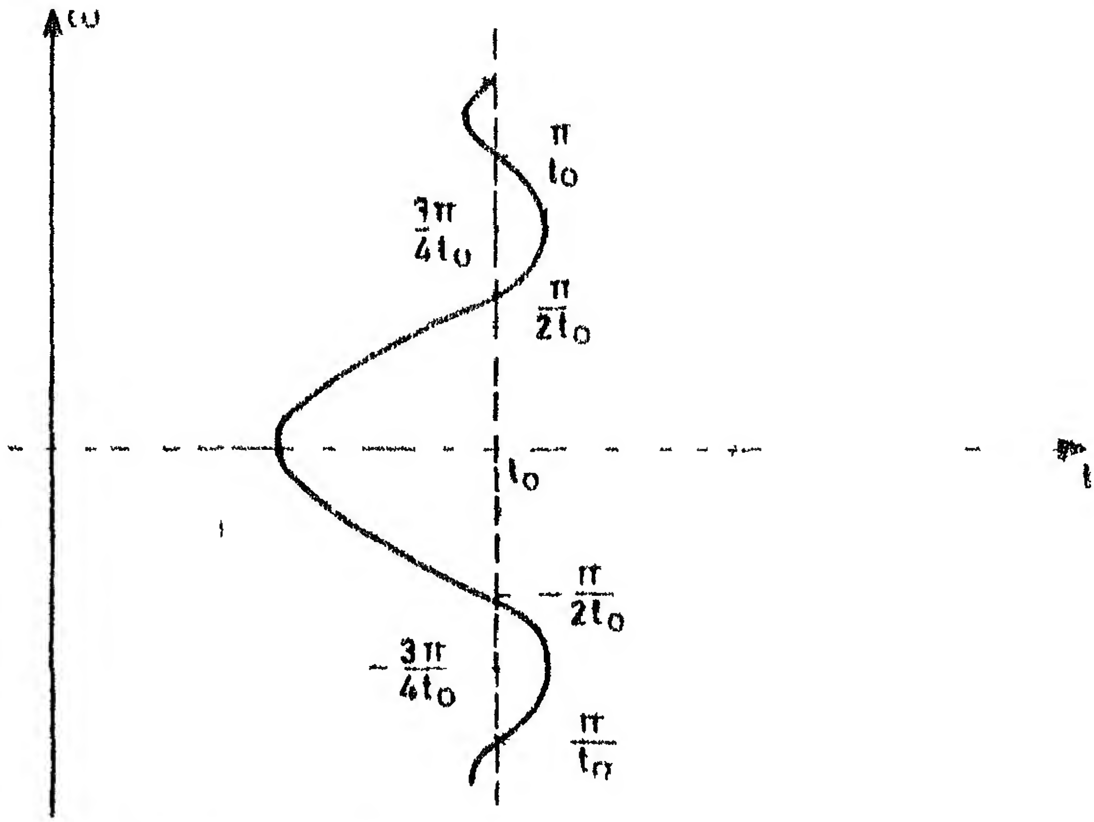


FIG 5 1(a) WD OF $h(t)$ AT $t = t_0$

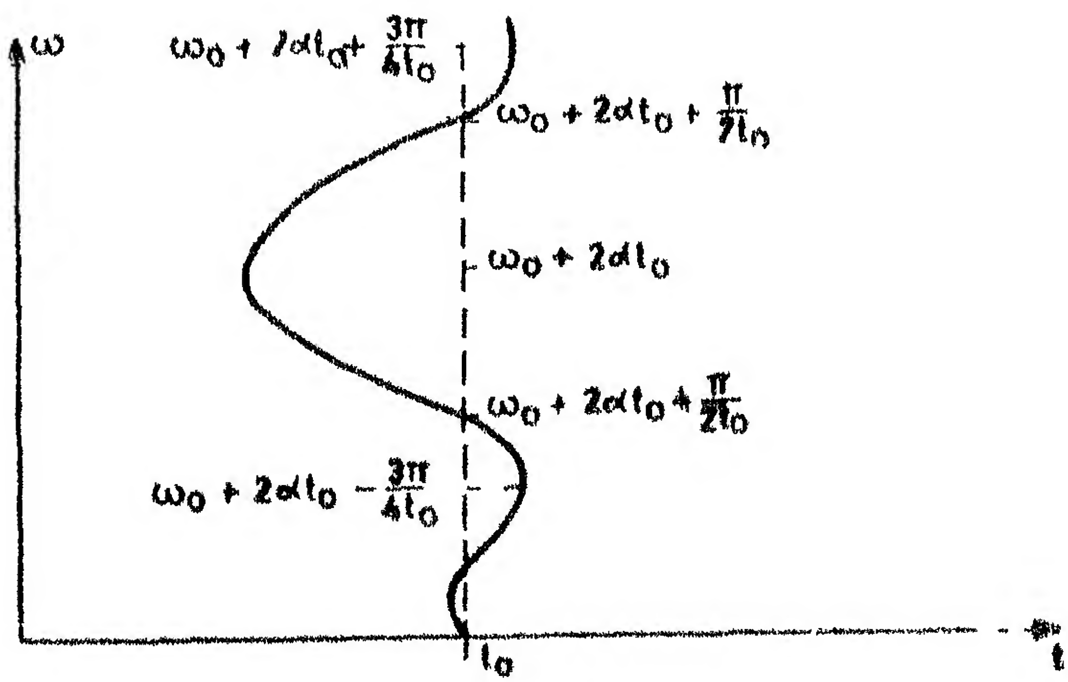


FIG 5 1(b) WD OF $f(t)$ AT $t = t_0$.

exactly like this except for the fact that at any time instant $t = t_o$, the entire distribution is shifted by $(\omega_o + 2\alpha t_o)$ along the positive ω-axis as shown in fig 5 1(b) Thus the central lobe of the WD of f(t) at time t will be at a frequency $\omega = (\omega_o + 2\alpha t)$ Also adding the distribution values of f(t) along line $\omega = \omega_o + 2\alpha t + b$ (where b is a constant), is equivalent to adding the distribution values of h(t) along a horizental line $\omega = b$ The WD of h(t) reveals that at any fixed time instant t ($\leq T_o/2$) the WD has central lobe at $\omega = 0$ and the first side lobe at $\omega = \frac{3\pi}{4t}$ (approximately) This means that if we add up the distribution values of h(t) along different horizental lines $\omega = b$, we expect a sharp peak at b = 0 and since the secondary side-lobes do not fall on same horizental line, we expect that as b increase (or decreases) the summation should decrease continuously without any prominent side-lobes. This, if it works, will be in contrast to the correlator (or matched filter) output which shows prominent side-lobes and hence can be used in a reception scheme With this motivation the following sections try to work out details of such a reception scheme and compare its performance with that of correlator receiver

5 3 RECEPTION SCHEME USING THE CORRELATOR

Let us for the moment assume that the received signal r(t) given by equation (5 1) consist of only one return with

unity reflection coefficient and time-delay τ. Also let the constant A of the equation (5.2) be such that the signal $f(t)$ will have energy E

$$A = \frac{E}{T_o} \tag{5.8}$$

Thus the received signal $r(t)$ becomes

$$r(t) = \frac{E}{T_o} e^{j\omega o(t-\tau)} e^{j\mu(t-\tau)^2} + n(t) \quad \text{for } \tau \le t \le (\tau + T_o)$$

$$= n(t) \text{ otherwise} \tag{5.9}$$

Our objective is to determine the time-delay τ. Since the noise is additive white gaussian, the correlator (or matched filter) reception is the best as far as SNR at the output is concerned. In this scheme we correlate the received signal with the reference signal $f_{t_o}(t) = f(t-t_o)$ and this correlation is carried out at all time instants t_o. Thus the output $C(t)$ of the correlator is given by

$$C(t) = \int_t^{t+T_o} r(\eta)\, f^*(\eta-t)\, d\eta \tag{5.10}$$

The output $C(t)$ consists of two parts

$$\text{Signal part } C_s(t) = \int_t^{t+T_o} f_\tau(\eta) f^*(\eta-t)\, d\eta \quad \text{for } (\tau-T_o) \le t \le (\tau+T_o)$$

$$= 0 \text{ otherwise} \tag{5.11}$$

and Noise Part $C_N(t) = \int_{t}^{t+T_o} n(\eta) f^*(\eta - t)\, d\eta$ (5 12)

The signal part $C_s(t)$ is maximum when $t = \tau$ and is equal to

$$C_s(t)_{max} = C_s(t - \tau) = E \qquad (5\ 13)$$

By observing the output $C(t)$ we wish to find out the value of τ Since the signal part of the output $C(t)$ is maximum and real at $t = \tau$, we must concentrate only on the real part of $C_s(t)$ Redefining $C_s(t)$ as the real part of the signal part of the output, we get

$$C_s(t) = R\,P \int_{t}^{t+T_o} f(\eta) f^*(\eta - t)\, d\eta \qquad \text{for } (\tau - T_o) \leq t \leq (\tau + T_o)$$

$$= 0 \text{ otherwise} \qquad (5\ 14)$$

where R P stands for 'Real Part of'. Since we are mainly interested in studying the side-lobe behaviour of this reception scheme we can assume

$$\tau = 0 \qquad (5\ 15)$$

By carrying out the integration of equation (5 14) it can be shown that

$$C_s(t) = \frac{E}{T_o \alpha t} \sin\{\alpha t (T_o - t)\} \cos\{\omega_o t + \alpha t T_o\}$$

$$\text{for } 0 \le t \le T_o$$

$$= \frac{E}{T_o \alpha t} \sin\{\alpha t (T_o + t)\} \cos(\omega_o t + \alpha t T_o)$$

$$\text{for } -T_o \le t \le 0$$

$$= 0 \text{ otherwise} \qquad (5\ 16)$$

The derivation of this result is given in Appendix A

The envelope $A_c(t)$ of the output $C_S(t)$ is given by

$$A_c(t) = \frac{E}{T_o \alpha t} \left| \sin\{\alpha t\ (T_o - t)\} \right| \text{ for } 0 \le t \le T_o$$

$$= \frac{E}{T_o \alpha |t|} \left| \sin\{\alpha t\ (T_o + t)\} \right| \text{ for } -T_o \le t \le 0$$

$$= 0 \text{ otherwise} \qquad (5\ 17)$$

The noise part $C_N(t)$ of the output is a stochastic process with gaussian distribution The variance σ^2 of this noise can be shown to be

$$\sigma^2 = \frac{EN}{2} \qquad (5\ 18)$$

where $N/2$ is the noise spectral density of the input noise $n(t)$ (see Appendix 1)

SNR at the output when the signal part of the output is maximum is given by

$$(SNR)_{correlator} = \frac{2E}{N} \tag{5 19}$$

We started this section with the assumption of only one return with unity reflection coefficient For a received signal consisting of many returns with different reflection coefficients as given by equation (5 1), the correlator output will be the superimposition of the outputs of the individual returns The outputs of the individual returns will have peaks corresponding to the delays of the returns and the amplitudes of the peaks will be proportional to the corresponding reflection coefficients However each of these outputs will have secondary side-lobes apart from the main lobe Hence the superimposition of the individual outputs will still have peaks corresponding to each return but the time instants of the peaks may not exactly match the delays of the returns Similarly the amplitudes of the peaks may not be exactly proportional to the reflection coefficients of the corresponding returns.

In section 5 6 simulation results of the correlator output have been presented assuming two returns with variable delays and reflection coefficients

5 4 RECEPTION SCHEME USING THE WIGNER DISTRIBUTION

The reception scheme using the WD consists of taking the cross WD of the received signal $r(t)$ and the reference signal $f_{t_0}(t) = f(t-t_0)$ and then summing up the distribution values along the line $\omega = \omega_0 + 2\alpha(t-t_0)$ of the $t-\omega$ plane This procedure is repeated for all time instants t_0 and the instants and amplitudes corresponding to the peaks in the output provide a measure of the reflection coefficients and the time delays As in section 5 3 we assume the received signal to be consisting of one return only with unity reflection coefficient and time-delay τ At the output of the receiver we expect a peak when $t_0 = \tau$ which should fall rapidly as t_0 increases further

The output $W(t)$ of the WD based receiver is given by

$$W(t) = \int_{t}^{t+T_0} W_{r,f_t} [\eta, \omega = \omega_0 + 2\alpha(\eta-t)] \, d\eta \qquad (5\ 20)$$

when $f_t(x) = f(x-t)$.

Using equation (5.9) we get

$$W(t) = \int_t^{t+T_o} W_{f+n,f_t}[\eta, \omega = \omega_o + 2\alpha(\eta - t)]d\eta$$

$$\text{for } (\tau - 2T_o) \le t \le (\tau + 2T_o)$$

$$= \int_t^{t+T_o} W_{n,f_t}[\eta, \omega = \omega_o + 2\alpha(\eta - t)]d\eta \quad \text{otherwise}$$

(5 21)

This can be written as

$$W(t) = \int_t^{t+T_o} W_{f,f_t}[\eta, \omega = \omega_o + 2\alpha(\eta - t)]d\eta + \int_t^{t+T_o} W_{n,f_t}[\eta, \omega =$$

$$\omega_o + 2\alpha(\eta - t)]d\eta \qquad \text{for } (\tau - 2T_o) \le t \le (\tau + 2T_o)$$

$$= \int_t^{t+T_o} W_{n,f_t}[\eta, \omega = \omega_o + 2\alpha(\eta - t)]d\eta \quad \text{otherwise}$$

. (5 22)

The output $W(t)$ consist of two parts

Signal Part $W_s(t) = \int_t^{t+T_o} W_{f,f_t}[\eta, \omega = \omega_o + 2\alpha(\eta - t)]d\eta$

$$\text{for } (\tau - 2T_o) \le t \le (\tau + 2T_o)$$

$$= 0 \text{ otherwise}$$

(5 23)

and Noise Part $\quad W_N(t) = \int_t^{t+T_o} W_{n,f_t}[\,\eta, \omega = \omega_o + 2a(\eta - t)\,]\, d\eta$

(5 24)

The signal part $W_s(t)$ is maximum when $t = \tau$ and is equal to

$$W_s(t)_{max} = W_s(t-\tau) = ET_o \qquad (5\ 25)$$

The derivation of this result will be presented shortly

Again since the maximum value of $W_s(t)$ in which we are interested, is real, we should concentrate only on the real part of $W_s(t)$ Redefining $W_s(t)$ as the real part of the signal part of the output, we get

$$W_s(t) = R\ P \int_t^{t+T_o} W_{f_\tau, f_t}[\,\eta,\ \omega = \omega_o + 2a(\eta - t)\,]\, d\eta$$

$$\text{for } (\tau - 2T_o) \leq t \leq (\tau + 2T_o)$$

$$= 0 \text{ otherwise} \qquad (5\ 26)$$

As is done in Section 5 2 let us assume $\tau = 0$ The value of $W_s(t)$ at $t = t_o$ can be obtained by first computing the WD, $W_{f,f_{t_o}}(t, \omega)$. Referring to Fig 5 2 and using the definition of the WD it can be seen that

$$W_{f,f_{t_o}}(t, \omega) = 0 \text{ for } t \leq t_o/2 \quad \text{or} \quad t \geq T_o + t_o/2 \qquad (5.27)$$

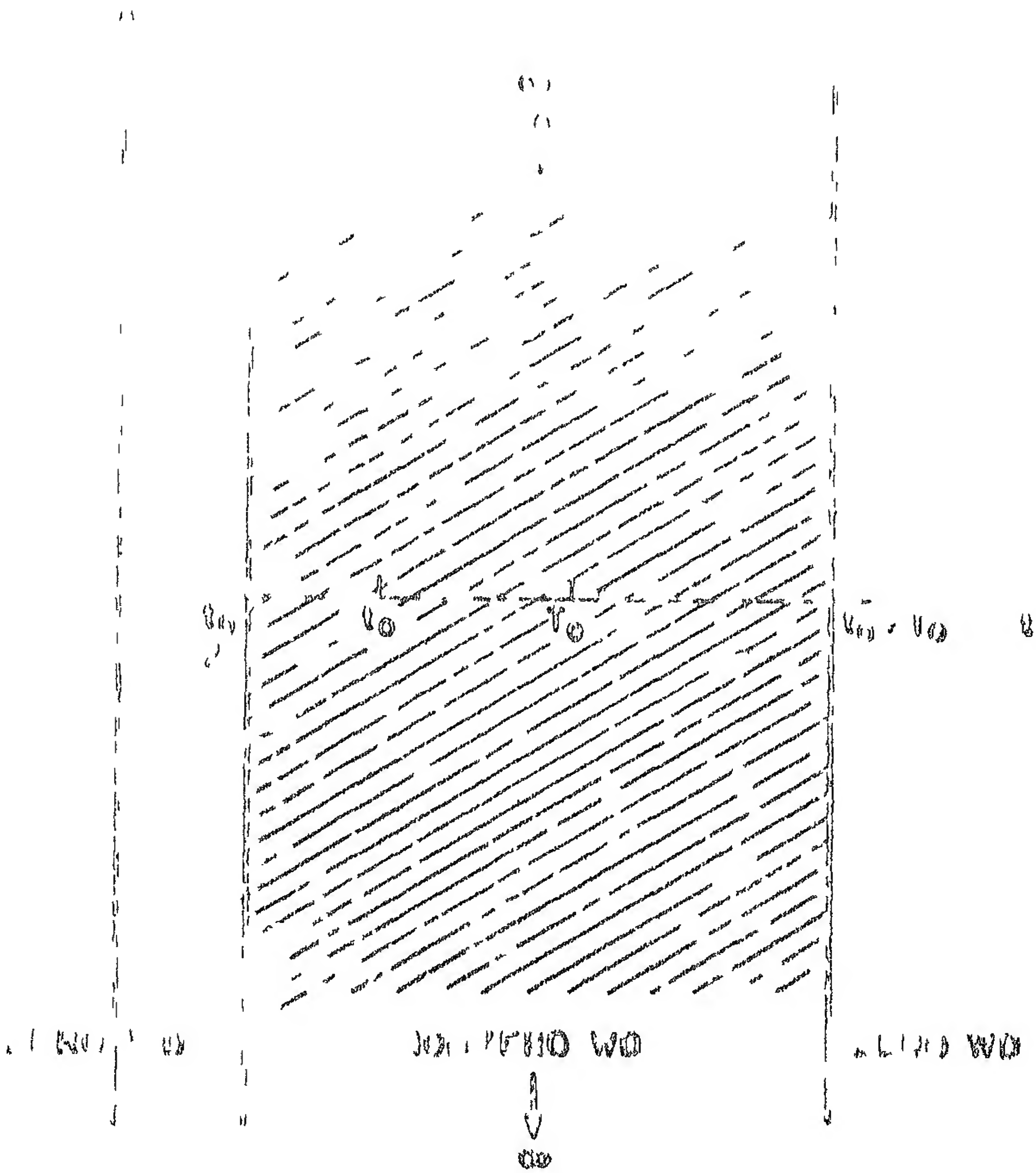


FIG. 5.[illegible] CROSS [illegible] OF $f(t)$ AND $v_m(t)$

For $t_o/2 \le t \le T_o + t_o/2$

$$W_{f,f_{t_o}}(t,\omega) = \int_{-\infty}^{+\infty} f(t+\frac{\eta}{2})\, f_{t_o}(t-\eta/2)\, e^{-j\omega\eta}\, d\eta$$

$$= \int_{-\infty}^{+\infty} f(t+\frac{\eta}{2})\, f^*(t - \frac{\eta}{2} - t_o)\, e^{-j\omega\eta}\, d\eta$$

Let $\eta = \beta - t_o$

$$W_{f,f_{t_o}}(t,\omega) = \int_{-\infty}^{+\infty} f(t - \frac{t_o}{2} + \frac{\beta}{2})\, f^*(t - \frac{t_o}{2} - \frac{\beta}{2})\, e^{-j\omega(\beta - t_o)}\, d\beta$$

$$= e^{j\omega_o t_o}\, W_{f,f}(t - \frac{t_o}{2}, \omega) \qquad \text{for } t_o/2 \le t \le T_o + t_o/2$$

(5 28)

Hence summation of the WD values $W_{f,f_{t_o}}(t,\omega)$ along an axis $\omega = \omega_o + 2a(t-t_o)$ is equivalent to the summation of the WD values $W_{f,f}(t,\omega)$ along an axis $\omega = \omega_o + 2a(t-t_o/2)$ with t going from $t_o/2$ to $T_o + t_o/2$ and with each WD value being multiplied by $e^{j\omega t_o}$ As has been explained in section 5 2, this in turn is equivalent to the summation of the WD values $W_{h,h}(t,\omega)$, defined in equation (5 7), along a horizontal line $\omega = -at_o$ with t going from $t_o/2$ to $T_o + \frac{t_o}{2}$ and with each WD value being multiplied by $e^{j\omega' t}$, where $\omega' = \omega_o + 2a(t-t_o/2)$

Hence

$$W_s(t) = R\,P \int_{t/2}^{T_o+t_o/2} W_{h,h}[x, \omega = -\alpha t]\, e^{j[\omega_o + 2\alpha(x-t/2)]t}\, dx \tag{5 29}$$

By using equation (5 7) and carrying out the above integration we get

$$W_s(t) = 0 \qquad \text{for } t \le -2T_o$$

$$= \left\{ \frac{E(2T_o+t)}{2T_o \alpha t} \sin(2\alpha t T_o - \alpha t^2) + \frac{E}{2T_o \alpha^2 t^2} \sin(\alpha t T_o) \sin(\alpha t T_o + \alpha t^2) \right\} \cos(\omega_o t)$$

$$+ \left\{ \frac{E(2T_o+t)}{2T_o \alpha t} \cos(2\alpha t T_o - \alpha t^2) - \frac{E}{2T_o \alpha^2 t^2} \cos(\alpha t T_o) \sin(\alpha t T_o + \alpha t^2) \right\} \sin(\omega_o t)$$

$$\text{for } -2T_o \le t \le -T_o$$

$$= \left\{ \frac{E}{\alpha t} \cos(\alpha t T_o - \alpha t^2) \sin(\alpha t T_o) + \frac{E}{2T_o \alpha} \sin(2\alpha t T_o - \alpha t^2) \right.$$

$$\left. - \frac{E}{2T_o \alpha^2 t^2} \sin(\alpha t^2) \sin(2\alpha t T_o) \right\} \cos(\omega_o t)$$

$$+ \left\{ \frac{E}{2T_o \alpha} \cos(2\alpha t T_o - \alpha t^2) - \frac{E}{2T_o \alpha^2 t^2} \sin(\alpha t^2) \cos(2\alpha t T_o) \right.$$

$$\left. - \frac{E}{\alpha t} \sin(\alpha t T_o - \alpha t^2) \sin(\alpha t T_o) \right\} \sin(\omega_o t)$$

$$\text{for } -T_o \le t \le 0$$

$$= \left\{ \frac{E(T_o - t)}{2T_o \alpha t} \sin(\alpha t^2) + \frac{E}{2\alpha t} \sin(2\alpha t T_o - \alpha t^2) \right\} \cos(\omega_o t)$$

$$+ \left\{ \frac{E}{\alpha t} \cos(2\alpha t T_o - \alpha t^2) - \frac{E}{2T_o \alpha^2 t^2} \sin \alpha t^2 - \frac{E(T_o - t)}{2T_o \alpha t} \cos(\alpha t^2) \right\}$$

$$\sin(\omega_o t)$$

$$\text{for } 0 \leq t \leq T_o$$

$$= \left\{ \frac{E(T_o - t)}{T_o \alpha t} \sin(2\alpha t T_o - \alpha t^2) - \frac{E}{2T_o \alpha^2 t^2} \sin(2\alpha t T_o - \alpha t^2) \right.$$

$$\left. \sin(\alpha t^2) \right\} \cos(\omega_o t)$$

$$+ \left\{ \frac{E(T_o - t)}{T_o \alpha t} \cos(2\alpha t T_o - \alpha t^2) - \frac{E}{2T_o \alpha^2 t^2} \sin(2\alpha t T_o - \alpha t^2) \right.$$

$$\left. \cos(\alpha t^2) \right\} \sin(\omega_o t)$$

$$\text{for } T_o \leq t \leq 2T_o$$

$$= 0 \text{ for } t \geq 2T_o \qquad (5\ 30)$$

Derivation of the above results is given in Appendix B It is interesting to note that $W_s(t)$ is not symmetric about the $W_s(t)$

axis The envelope of the output can be obtained by squaring the coefficients of $\cos(\omega_o t)$ and $\sin(\omega_o t)$ terms adding them and taking the square root The value of $W_s(t)$ at $t = 0$ is maximum and equals $E\Gamma_o$ as mentioned in equation (5 25).

The noise part of the output $W_N(t)$ is a stochastic process with gaussian distribution The variance of this noise can be obtained by applying the definition of the variance to equation (5 25) Rewritting equation (5 25) we get

$$W_N(t) = \int_t^{t+T_o} W_{n,f_t}\,[\,\eta, \omega = \omega_o + 2\alpha\,(\eta - t)\,]\,d\eta$$

Let $x = \eta - t$

$$W_N(t) = \int_0^{T_o} W_{n,f_t}\,[\,t+x,\ \omega = \omega_o + 2\alpha x\,]\ dx$$

$$= \int_{x=0}^{T_o} \int_{\eta=-\infty}^{+\infty} n(t+x\,\tfrac{\eta}{2})\,f^*(x-\tfrac{\eta}{2})\ e^{-j2\alpha x\eta}\,e^{-j\omega_o\eta}\ d\eta\,dx \qquad (5\ 31)$$

$$E\,[\,W_N(t_1)\ W_N^*(t_2)\,] = E \int_{x=0}^{T_o} \int_{y=0}^{T_o} \int_{\eta=-\infty}^{+\infty} \int_{\beta=-\infty}^{+\infty} n(t_1+x+\tfrac{\eta}{2})\,n^*(t_2+y\ \beta/2)$$

$$f\ (x-\eta/2)\ f(y-\beta/2)\,.\,e^{-j\omega_o\eta}\ e^{j\omega_o\beta}\ e^{-j2\alpha x\eta}\,e^{j2\alpha y\beta}\ d\beta\,d\eta\,dy\,dx$$

Since $n(t)$ is white noise with spectral density $N/2$ we have

$$E\,[n(t_i)\,n^*(t_j)\,] = \frac{N}{2}\,\delta(t_i-t_j) \tag{5.32}$$

$$E\,[W_N(t_1)\,W_N^*(t_2)\,] = \int_{x=0}^{T_o}\int_{y=0}^{T_o}\int_{\eta=-\infty}^{+\infty}\int_{\beta=-\infty}^{+\infty} \frac{N}{2}\,\delta\,(t_1+x+\frac{\eta}{2}-t_2-y-\frac{\beta}{2})$$

$$f^*(x-\eta/2)\,f(y-\beta/2)\;e^{-j2\alpha x\eta}\,e^{+j2\alpha y\beta}\,e^{-j\omega_o\eta}\,e^{j\omega_o\beta}\;d\beta\, d\eta\, dy\, dx$$

$$= \int_{x=0}^{T_o}\int_{y=0}^{T_o}\int_{\eta=-\infty}^{+\infty} \frac{N}{2}\, f^*(x-\frac{\eta}{2})\,f(2y+t_2-t_1-\frac{\eta}{2}-x)\;e^{-j\omega_o\eta}\,e^{j\omega_o(2t_1+2x+\eta-2t_2-2y)}$$

$$e^{-j2\alpha x\eta}\,e^{j2\alpha y(2t_1+2x+\eta-2t_2-2y)}\;d\eta\, dy\, dx$$

Now for given x, y, t_1, t_2 there exist values of η such that $(x-\frac{\eta}{2})$ and $(2y+t_2-t_1-x-\eta/2)$ both lie in the interval $[\,0,\ T_o]$. Let R be that range of η. Hence whenever η belongs to R we can substitute the expression of the function f given by equation (5.2)

$$\therefore\; E\,[W_N(t_1)\,W^*(t_2)] = \int_{x=0}^{T_o}\int_{y=0}^{T_o}\int_{\eta\in R} \frac{EN}{T_o}\, e^{-j\omega_o(x-\eta/2)}\; e^{-j\alpha(x-\eta/2)^2}$$

$$e^{j\omega_o(2y+t_2-t_1-\eta/2-x)}\; e^{j\omega_o(2t_1+2x-2t_2-2y)}\; e^{j\alpha(2y+t_2-t_1-\eta/2-x)^2}$$

$$e^{-j2\alpha x\eta}\; e^{j2\alpha y(2t_1+2x+\eta-2t_2-2y)}\; d\eta\, dy\, dx$$

$$= \int_{x=0}^{T_0} \int_{y=0}^{T_0} \int_{\eta \in R} \frac{EN}{T_0} \, e^{j\alpha[(t_2-t_1)^2-2x(t_2-t_1)-\eta(t_2-t_1)]} \, e^{-j\omega_0(t_2-t_1)} \, d\eta\, dy\, dx \qquad (5.33)$$

Thus $E[W_N(t_1)W_N^*(t_2)]$ is a function of (t_2-t_1) only. Hence the noise $W_N(t)$ is stationary as expected. To compute the variance of $W_N(t)$ put $t_1 = t_2$ in equation (5.33)

$$\therefore E[W_N(t)\, W_N^*(t)] = \int_{x=0}^{T_0} \int_{y=0}^{T_0} \int_{\eta \in R} \frac{EN}{T_0}\, d\eta\, dy\, dx \qquad (5.34)$$

The region R can be found by separating the intervals of x and y into smaller intervals and then carrying out the triple integration of equation (5.34). This can be done and the variance σ^2 of the noise $W_N(t)$ can be shown to be equal to

$$\sigma^2 = \frac{5ENT_0^2}{6} \qquad (5.35)$$

Derivation of this result is given in Appendix C.

∴ The SNR for maximum output

$$= \frac{(W_s(t)_{max})^2}{\sigma^2}$$

$$(SNR)_{WD} = \frac{6L^2T_o^2}{5ENT_o^2} = \frac{6}{5}\frac{E}{N} \tag{5 36}$$

Comparing this value with the SNR value of the correlator receiver (equation (5 19)), we see that there is a degradation by a factor of 0 6 or a degradation of -2 2184 dB as far as SNR performance of the WD based receiver is concerned

For received signal consisting of more than one returns the output will be the superimposition of the outputs of the individual returns As in the case of the correlator receiver output, this superimposition also will have peaks corresponding to each return but the delays and the amplitudes of these peaks may not be an exact measure of the delays and the reflection coefficients of the returns Section 5 6 presents some of the simulation results assuming two returns with variable delays and reflection coefficients.

5 5 WD BASED RECEIVER AS AN EQUIVALENT LINEAR FILTER

From equation (5.20), which gives the output of the WD based receiver, it is clear that this reception is linear and time-invariant in nature Hence it can be replaced by an equivalent linear time-invariant filter

The impulse response of such a filter can be obtained by considering

$$r(t) = \delta(t) \qquad (5\ 37)$$

in equation (5 20) It can be shown that the required impulse response $h_w(t)$ is given by

$$
\begin{aligned}
h_w(t) &= 0 & -2T_o &\geq t \\
&= \sqrt{\frac{E}{T_o}}\,(2T_o + t)\, e^{j\omega_o t}\, e^{-j\alpha t^2} & -2T_o &\leq t \leq -T_o \\
&= \sqrt{E\,T_o}\;\; e^{j\omega_o t}\;\; e^{-j\alpha t^2} & -T_o &\leq t \leq 0 \\
&= \sqrt{\frac{E}{T_o}}\,(T_o - t)\, e^{j\omega_o t} e^{-j\alpha t^2} & 0 &\leq t \leq T_o \\
&= 0 & t &\geq T_o \qquad (5\ 38)
\end{aligned}
$$

Derivation of this result is included in Appendix D

The impulse response of the correlator receiver $h_c(t)$, is given by

$$
\begin{aligned}
h_c(t) &= 0 & -T_o &\geq t \\
&= \sqrt{\frac{E}{T_o}}\, e^{-j\omega_o t}\quad e^{-j\alpha t^2} & -T_o &\leq t \leq 0 \\
&= 0 & t &\geq 0
\end{aligned}
$$

$$(5\ 39)$$

Comparing the two impulse responses $h_w(t)$ and $h_c(t)$ it is seen that in the interval $-T_o \leq t \leq 0$, both have exactly same value except for a scaling factor of T_o and sign change of the carrier frequency term Outside this interval $h_c(t)$ is zero whereas $h_w(t)$ is still nonzero on both sides upto a length of T_o

The output of the WD based receiver now can be written as

$$W(t) = r(t) * h_w(t) \tag{5 40}$$

As has been mentioned before for the case of single return with zero delay and unity reflection coefficient, the peak of the output occurs at $t = 0$ in both the receivers In the correlator receiver both the signal part and the noise part of the output get affected by the entire impulse response $h_c(t)$. However in the WD based receiver the signal part of the output is due to only a portion of the impulse response, viz impulse response in the range $-T_o \leq t \leq 0$, whereas the noise is affected by the entire impulse response $h_w(t)$ This explains the degradation of SNR at the output of the WD based receiver compared to the correlator receiver as mentioned in section 5.4 At the same time it is precisely this 'extra'

portion of the impulse response $h_W(t)$, which is expected to suppress the sidelobes in the output of the WD based receiver

It is now clear that the WD based receiver, although defined by equation (5 20), need not be implemented directly as per this equation because this involves computation of a large number of WD values which is inherently a complex and slow process Instead implementation using the impulse response $h_W(t)$, is more convenient from a practical point of view

5 6 SIDE-LOBES OF THE CORRELATOR RECEIVER AND THE WD BASED RECEIVER

In this section simulation results of the outputs of the correlator receiver and the WD based receiver have been presented Since we are interested mainly in the side-lobe characteristics of the output, the noise has been omitted from the received signal r(t)

For a single return with unity reflection coefficient the envelope of the output of the correlator receiver is given by equation (5 17) The envelope of the WD based receiver can be obtained from equation (5 30) Both these envelopes have been plotted for different values of the parameters α and T_o. The plots have been presented in Appendix D

It can be seen from the plots with fixed $T_o = 2$ Secs and variable α that for small values of α the widths of the mainlobe and the first sidelobe are small in the output of the WD based receiver. The maximum amplitudes of the sidelobes are however more or less same in both the outputs and hence there is no suppression of the sidelobes in the WD based receiver. Moreover as α increases even this marginal improvement in the lobe-width reduction vanishes and both the outputs have similar sidelobe characteristics.

A better way to compare the two reception schemes for the sidelobe performance is to actually consider two or more returns with different reflection coefficients. For the purpose of simulation, two returns have been assumed. The first return has zero delay and unity reflection coefficient. The delay τ_2 and the reflection coefficient a_2, of the second return are variables. The plots of the envelopes of the outputs for different values of a_2 and τ_2 have been given in Appendix F.

It can be seen from these plots that for $a_2=1\ (=a_1)$, both the outputs have similar nature for all values of τ_2. Both the receivers are unable to detect the second return clearly for small τ_2 and can detect it with same clarity for

large τ_2 For smaller values of a_2 (a_2=0.1,0 6), the performance however differs Here again when τ_2 is smaller than the width of the mainlobe, both the receivers fail to detect the second return. But if τ_2 is such that the second return falls in the first sidelobe of the output of the first return (τ_2=40,50), the WD based receiver has better clarity or the discriminating ability, i e the slope of the rise and the magnitude of the rise of the peak in the output corresponding to the second return are both large in the WD based receiver When the second return falls in the second sidelobe of the output (τ_2 = 70,80), similar result is observed In this case the discriminating ability of the WD based receiver is more pronounced but at the same time there is a presence of a small spurious lobe in between the two lobes (corresponding to the two returns), which is absent in the correlator receiver output

When a_2 is further reduced to 0.2, similar results are observed. When the second return lies in the second sidelobe of the output of the first return, the correlator receiver is almost unable to detect it WD based receiver, however can detect it clearly although the amplitude of this second peak is more than proportional to a_2.

Overall it is seen that both the reception schemes have similar performance for determining the values of the time delays In the determination of the reflection coefficient values also, both perform equally well (or bad), in an overall sense, i.e. in certain cases WD based receiver gives better results and in others the correlator receiver However as far as the detection of the presence of the second return with smaller reflection coefficient is concerned, the WD based receiver has distinctly better performance This is particularly true when the second return lies in the first or the second sidelobe of the output of the first return.

CHAPTER 6

CONCLUSIONS

This chapter is aimed at reviewing the significant results obtained during the course of this work and making a few suggestions for the future line of work in the application areas of the WD

A combined time-frequency representation of signals and systems is an extremely useful tool in signal processing This is particularly so for nonstationary signals whose frequency content changes with time The example given in Chapter 1 clearly indicates the shortcomings of the conventional representations and brings forth the usefulness of time-frequency representations A general class of such representations was proposed by Cohen and has been discussed in Chapter 2 WD is a member of this class with properties more suitable for applications in signal processing

Definition of the WD is similar to that of the Ambiguity Function and hence most of the later's ground work can straightaway be extended to the WD Chapter 3 gives an exhaustive list of the mathematical properties of the WD

Most (but not all) of the results look similar to those of the Ambiguity Function but details differ. A method for fast computation of the WD has also been developed in this chapter

WD also finds its usefulness in optical signal processing. Examples given in Chapter 4 are only a small glimpse of the whole application area currently under active consideration. Methods for optical generation of the WD have been suggested [5], [6]. Input output relationships are simplified in the WD domain in many of the commonly used optical systems like lenses, magnifiers, fourier transformers etc. Study of the diffraction patterns in a complex environment will also be simplified in the WD domain

A reception scheme based on the WD is proposed in Chapter 5 to overcome the problem of side-lobe interference. Details of this new scheme for the reception of pulsed chirp signal have been worked out. Compared to the correlator receiver this receiver has a degradation of -2 2184dB as far as the SNR at the output is concerned. The side-lobe performance however shows some improvement though not as much as was thought of. For the computation of time-delays and

reflection coefficients there is no marked improvement in the new receiver but from the point of view of detection of weaker returns falling in the side-lobes of stronger returns it clearly shows advantage over the correlator receiver The peak in the output corresponding to such a weak return is more sharp in the WD based receiver indicating its better discriminating ability There are cases when the correlator receiver completely fails to detect a weaker return whereas the WD based receiver does it comfortably

Some recent developments in the field of WD, which have not been touched so far in this thesis, deserve a mention at this stage. WD has been used successfully in the design of loudspeakers by Janse and Kaizer[15]opening a vast new area of applications in acoustics A systematic method of the synthesis of a signal from its WD has been given by Kai-Bar Yu[38] Claasen and Mecklenboker have discussed in detail aliasing in the discrete WD [39] A beginning has been made by Kay and Boudreaux-Bartels in using the WD for detection problem [40] . Approach used by them is quite similar to the one discussed in Chapter 5

FUTURE SCOPE OF WORK

Any analysis involving nonstationarity merits an attention from the WD Speech analysis is one such example The spectrograms used extensively in speech analysis are based on the assumption of stationarity over the window length. Such an artificial assumption will not be required to be made while using the WD Moreover the time and frequency resolutions in the spectrogram should both be better with the use of WD However WD based spectrograms will need huge amount of computations and fast computational methods must be developed One such method has been discussed in section 3 7 but it needs to be improved further

The WD based receiver discussed in Chapter 5 assumes only time delay in the returns but no doppler shift An extention of the same reception scheme can be made to accomodate this also Such a modification will involve summing up of the distribution values along lines with different slopes in the $t-\omega$ plane and looking for the peaks It will be interesting to investigate the SNR and the side-lobe characteristics of such a receiver

APPENDIX A

A 1 SIGNAL PART OF THE OUTPUT OF THE CORRELATOR RECEIVER FOR A SINGLE RETURN

According to equation (5 14) the signal part of the output of the correlator receiver for a single return with zero time delay is

$$C_s(t) = R\,P \int_t^{t+T_o} f(\eta) f^*(\eta - t)\,d\eta \qquad \text{for } -T_o \le t \le T_o$$

$$= 0 \text{ otherwise}$$

$f(\eta)$ becomes zero for $\eta \le 0$ and $\eta \ge T_o$

$\therefore$ For $0 \le t \le T_o$ we have

$$C_s(t) = R\,P \int_t^{T_o} f(\eta)\; f^*(\eta - t)\,d\eta$$

$$= R.P \int_t^{T_o} \frac{E}{T_o}\, e^{j\omega_o \eta}\; e^{ja\eta^2}\; e^{-j\omega_o(\eta - t)}\; e^{-ja(\eta - t)^2}\,d\eta$$

$$= R\,P. \int_t^{T_o} \frac{E}{t_o}\, e^{j\omega_o t}\; e^{jat(2\eta - t)}\,d\eta$$

$$= \frac{E}{T_o} \int_t^{T_o} \cos\{\omega_o t + \alpha t(2\eta - t)\}\, d\eta$$

$$= \frac{E}{2T_o \alpha t} \left[\sin\{\omega_o t + \alpha t\,(2T_o - t)\} - \sin(\omega_o t + \alpha t^2) \right]$$

$$= \frac{E}{T_o \alpha t} \sin\{\alpha t(T_o - t)\} \quad \cos(\omega_o t + \alpha t T_o)$$

Similarly for $-T_o \leq t \leq 0$ we have

$$C_s(t) = R\,P \int_t^{t+T_o} f(\eta) f^*(\eta - t)\, d\eta$$

$$= R\,P. \int_0^{t+T_o} e^{j\omega_o \eta}\, e^{j\alpha\eta^2}\, e^{-j\omega_o(\eta - t)}\, e^{-\alpha(\eta - t)^2}\, d\eta$$

$$= \frac{E}{T_o} \int_t^{t+T_o} \cos\{\omega_o t + \alpha t(2\eta - t)\} \quad d\eta$$

$$= \frac{E}{2T_o \alpha t} \left[\sin\{\omega_o t + \alpha t(2T_o + t)\} - \sin(\omega_o t - \alpha t^2) \right]$$

$$= \frac{E}{T_o \alpha t} \sin\{\alpha t(T_o + t)\} \cos(\omega_o t + \alpha t T_o)$$

A 2 VARIANCE OF THE NOISE PART OF THE OUTPUT OF CORRELATOR RECEIVER

The correlator receiver is a linear time-invariant filter with impulse response $h_c(t)$, given by equation (5 39)

Since the input noise is white with spectral density N/2 we have

$$E[\,|C_N(t)|^2\,] = \sigma^2 = \frac{N}{2}\int_{-\infty}^{+\infty} |h_c(t)|^2\,dt \tag{A 1}$$

$$= \frac{EN}{2T_o}\int_{-T_o}^{0} dt$$

$$= \frac{EN}{2}$$

APPENDIX B

SIGNAL PART OF THE OUTPUT OF WD BASED RECEIVER FOR A SINGLE RETURN

Signal part $W_s(t)$ of the output of the WD based receiver for a single return with zero time delay is given by equation (5 29)

$$W_s(t) = R P \int_{t/2}^{T_o+t/2} W_{h,h}(x, \omega = -\alpha t) e^{j\{\omega_o + 2\alpha(x-t/2)\} t} dx$$

$$\text{for } -2T_o \leq t \leq 2T_o$$

$$= 0 \text{ otherwise} \qquad (B\ 1)$$

For the sake of computation of the above integration the interval $-2T_o \leq t \leq 2T_o$ would be divided into smaller sub-intervals

<u>Case 1</u>

$$-2T_o \leq t \leq -T_o$$

Since $W_{h,h}(t, \omega)$ is nonzero only for $0 \leq t \leq T_o$, the limits of the integration in equation (B 1) become zero to $(T_o+t/2)$ Substituting the value of $W_{h,h}(t,\omega)$ from equation (5 7) and noting that $T_o+t/2 \leq T_o/2$, we get

$$W_s(t) = \mathrm{Re} \int_0^{T_0+t/2} \frac{4Ex \sin(2\alpha t x)}{2T_0\, \alpha t x}\, e^{j\{\omega_0+2\alpha(x-t/2)\}\, t}\, dx$$

$$= \int_0^{T_0+t/2} \frac{2E}{T_0 \alpha t} \sin(2\alpha t x) \cos\{\omega_0 t + 2\alpha(x-t/2)t\}\, dx$$

$$= \frac{E}{T_0 \alpha t} \int_0^{T_0+t/2} \{\sin(\omega_0 t + 4\alpha t x - \alpha t^2) + \sin(-\omega_0 t + \alpha t^2)\}\, dx$$

$$= \frac{E}{T_0 \alpha t}\Big[(T_0+\tfrac{t}{2}) \sin(\omega_0 t + 2\alpha t T_0 - \alpha t^2) + \frac{1}{4\alpha t}\{\cos(-\omega_0 t - \alpha t^2)$$

$$- \cos(-\omega_0 t + 2\alpha t T_0 + \alpha t^2)\}\Big]$$

$$= \frac{E}{T_0 \alpha t}\Big[(T_0+\tfrac{t}{2}) \sin(\omega_0 t + 2\alpha t T_0 - \alpha t^2) + \frac{1}{2\alpha t}\{\sin(-\omega_0 t + \alpha t T_0)$$

$$\sin(\alpha t T_0 + \alpha t^2)\}\Big]$$

$$= \Big\{\frac{E(2T_0+t)}{2T_0 \alpha t} \sin(2\alpha t T_0 - \alpha t^2) + \frac{E}{2T_0 \alpha^2 t^2} \sin(\alpha t T_0)$$

$$\sin(\alpha t T_0 + \alpha t^2)\Big\} \cos(\omega_0 t)$$

$$+\left\{\frac{E(2T_o+t)}{2T_o\alpha t}\cos(2\alpha tT_o-\alpha t^2)-\frac{E}{2T_o\alpha^2t^2}\cos(\alpha tT_o)\sin(\alpha tT_o+\alpha t^2)\right\}$$

$$\cdot\ \sin(\omega_o t)$$

Case 2

$$-T_o \leq t \leq 0$$

Here the limits of the integration in equation (B 1) become zero to $(T_o + t/2)$ as in the previous case. But since in this interval $(T_o+t/2) \geq T_o/2$, we get

$$W_s(t) = R.P \int_0^{T_o/2} \frac{4Ex\ \sin(2\alpha tx)}{2T_o\alpha tx}\ e^{-j\{\omega_o+2\alpha(x-t/2)\}\ t}\ dx$$

$$+\ R\ P.\int_{T_o/2}^{T_o+t/2} \frac{4E(T_o-x)\sin\ 2\alpha t(T_o-x)}{2T_o\alpha t(T_o-x)}\ e^{j\{\omega_o+2\alpha(x-t/2)\}}\ dx$$

$$=\frac{E}{T_o\alpha t}\int_0^{T_o/2}\{\sin(\omega_o t+4\alpha tx-\alpha t^2)+\sin(-\omega_o t+\alpha t^2)\}\ dx$$

$$+\frac{E}{T_0\alpha t}\int_{T_0/2}^{T_0+t/2}\{\sin(\omega_0 t+2\alpha t x-\alpha t^2)\ \sin(-\omega_0 t+2\alpha t T_0-4\alpha t x+\alpha t^2)\}\,dx$$

$$=\frac{E}{T_0\alpha t}\Big[\frac{1}{4\alpha t}\{\cos(\omega_0 t-\alpha t^2)-\cos(\omega_0 t+2\alpha t T_0-\alpha t^2)\}$$

$$+\frac{T_0}{2}\sin(-\omega_0 t+\alpha t^2)\Big]+\frac{E}{T_0\alpha t}\Big[\left(\frac{T_0+t}{2}\right)\sin(\omega_0 t+2\alpha t T_0-\alpha t^2)$$

$$+\frac{1}{4\alpha t}\{\cos(-\omega_0 t-2\alpha t T_0-\alpha t^2)-\cos(-\omega_0 t+\alpha t^2)\}\Big]$$

$$=\frac{E}{T_0\alpha t}\Big[-\frac{1}{2\alpha t}\sin(\alpha t^2)\sin(\omega_0 t+2\alpha t T_0)+T_0\ \sin(\alpha t T_0)$$

$$\cos(\omega_0 t+\alpha t T_0-\alpha t^2)+\frac{t}{2}\sin(\omega_0 t+2\alpha t T_0-\alpha t^2)\Big]$$

$$-\{\frac{E}{\alpha t}\cos(\alpha t T_0-\alpha t^2)\sin(\alpha t T_0)+\frac{E}{2T_0\alpha}\sin(2\alpha t T_0-\alpha t^2)$$

$$-\frac{E}{2T_0\alpha^2 t^2}\sin(\alpha t^2)\sin(2\alpha t T_0)\}\cos(\omega_0 t)$$

$$+\{\frac{E}{2T_0\alpha}\cos(2\alpha t T_0-\alpha t^2)-\frac{E}{2T_0\alpha^2 t^2}\sin(\alpha t^2)\cos(2\alpha t T_0)$$

$$-\frac{E}{\alpha t}\sin(\alpha t T_0-\alpha t^2)\sin(\alpha t T_0)\}\sin(\omega_0 t)$$

Case 3

$$0 \le t \le T_o$$

Here the limits of integration become $t/2$ to T_o. Also $t/2 \le T_o/2$

$$W_s(t) = R\,P \int_{t/2}^{T_o/2} \frac{4Ex \sin(2\alpha tx)}{2T_o \alpha tx} e^{j\{\omega_o + 2\alpha(x - t/2)\} t} \, dx$$

$$+ R\,P \int_{T_o/2}^{T_o} \frac{4E(T_o - x) \sin 2\alpha t(T_o - x)}{2T_o \alpha t(T_o - x)} e^{j\{\omega_o + 2\alpha(x - t/2)\} t} \, dx$$

$$= \frac{E}{T_o \alpha t} \int_{t/2}^{T_o/2} \{\sin(\omega_o t + 4\alpha tx - 1\alpha t^2) \sin(-\omega_o t + \alpha t^2)\} \, dx$$

$$+ \frac{E}{T_o \alpha t} \int_{T_o/2}^{T_o} \{\sin(\omega_o t + 2\alpha t T_o - \alpha t^2) + \sin(-\omega_o t + 2\alpha t T_o - 1\alpha tx + \alpha t^2)\} \, dx$$

$$= \frac{E}{T_o \alpha t} \left[\frac{1}{4\alpha t} \{\cos(\omega_o t + \alpha t^2) - \cos(\omega_o t + 2\alpha t T_o - \alpha t^2)\} \right.$$

$$\left. + \left(\frac{T_o - t}{2}\right) \sin(-\omega_o t + \alpha t^2)\right] + \frac{E}{T_o \alpha t} \left[\frac{T_o}{2} \sin(\omega_o t + 2\alpha t T_o - \alpha t^2) \right.$$

$$\left. \frac{1}{4\alpha t} \{\cos(-\omega_o t + \alpha t^2 - 2\alpha t T_o) - \cos(-\omega_o t + \alpha t^2)\}\right]$$

$$= \frac{E}{T_0 \alpha t} \left[-\frac{1}{4\alpha t} \{\cos(\omega_0 t + \alpha t^2) - \cos(-\omega_0 t + \alpha t^2)\} + \left(\frac{T_0 - t}{2}\right) \sin(-\omega_0 t + \alpha t^2) \right.$$

$$\left. + \frac{T_0}{2} \sin(\omega_0 t + 2\alpha t T_0 - \alpha t^2) \right]$$

$$= \frac{E}{T_0 \alpha t} \left[-\frac{1}{2\alpha t} \sin(\alpha t^2) \sin(-\omega_0 t) + \left(\frac{T_0 - t}{2}\right) \sin(-\omega_0 t + \alpha t^2) \right.$$

$$\left. + \frac{T_0}{2} \sin(\omega_0 t + 2\alpha t T_0 - \alpha t^2) \right]$$

$$- \left\{ \frac{E(T_0 - t)}{2T_0 \alpha t} \sin(\alpha t^2) + \frac{E}{2\alpha t} \sin(2\alpha t T_0 - \alpha t^2) \right\} \cos(\omega_0 t)$$

$$+ \left\{ \frac{E}{\alpha t} \cos(2\alpha t T_0 - \alpha t^2) - \frac{E}{2T_0^2 \alpha^2 t^2} \sin \alpha t^2 \right.$$

$$\left. - \frac{E(T_0 - t)}{2T_0 \alpha t} \cos(\alpha t^2) \right\} \sin(\omega_0 t)$$

Case 4

$$T_0 \leq t \leq 2T_0$$

Here the limit of the integration is $t/2$ to T_0 and $t/2 \geq T_0/2$

$$\therefore \; W_s(t) = R\,P \int_{t/2}^{T_0} \frac{4E(T_0 - x)}{2T_0\, tx} \sin\{2\alpha t(T_0 - x)\}\, e^{-j\{\omega_0 t 2\alpha(x - t/2)\}\, t}\, dx$$

$$= \frac{E}{T_o \omega t} \int_{t/2}^{T_o} \{\sin(\omega_o t+2\ tT_o-\omega t^2)\ \sin(-\omega_o t+2\ tT_o-4\ tx+\ t^2)\}\ dx$$

$$= \frac{E}{T_o \omega t}\ [\ (T_o-t)\ \sin\ (\omega_o t+2\ tT_o-\ t^2)\ ]$$

$$+ \frac{E}{T_o \omega t}\ [\cos(-\omega_o t-2\omega tT_o+\ t^2)-\cos(-\omega_o t+2\ tT_o-3\ t^2)\ ]$$

$$= \frac{E}{T_o\ t}\ [\ (T_o-t)\sin(\omega_o t+2\ tT_o-\omega t^2)+2\sin(-\omega_o t-\ t^2)\sin(2\ tT_o-2\ t^2)]$$

$$=\{\ \frac{E(T_o-t)}{T_o \omega t}\ \sin(2\ tT_o-\ t^2)\ -\ \frac{E}{2T_o \omega^2 t^2}\ \sin(2\ tT_o-\ t^2)\sin(\ t^2)\ \}$$

$$\cos(\ \omega_o t)$$

$$+\{\frac{E(T_o-t)}{T_o \omega t}\ \cos(2\ tT_o-\ t^2)-\ \frac{E}{2T_o \omega^2 t^2}\ \sin(2\ tT_o-\ t^2)\cos(\ t^2)\ \}$$

$$\sin(\ \omega_o t)$$

APPENDIX C

VARIANCE OF THE NOISE PART OF THE OUTPUT OF WD BASED RECEIVER

As has been discussed in section 5 4, the variance of the output noise in the WD based receiver can be computed by carrying out the integration of equation (5 34) The range R of the permitted values of η can be found by dividing the intervals $x = 0$ to l_o and $y = 0$ to T_o into appropriate smaller subintervals Thus the double integration of equation (5.34) can be split into a number of double integrations, each of which will have range R defined in terms of x and y Computation of these smaller intervals is straightforward since the integrand of equation (5 34) is a just a constant

The procedure outlined above is quite lengthy and hence will not be included here although it has been verified that this leads to the value of variance given in equation (5 35) Instead an alternate derivation based as the impulse response concept is given below

As has been mentioned in Section 5 5, the WD based receiver can be replaced by an equivalent linear time-invariant filter with impulse response $h_w(t)$, given by equation

(5 38) Hence using the formula given by equation (A 1) we get

$$E\left[W_N(t)^2\right]\sigma^2 = \frac{N}{2}\int_{-\infty}^{+\infty} h_w(t)^2 \, dt$$

$$= \frac{NE}{2T_o}\int_{-2T_o}^{-T_o} (2T_o+t)^2 dt + \frac{NE}{2T_o}\int_{-T_o}^{0} T_o^2 \, dt$$

$$+ \frac{NE}{2T_o}\int_{0}^{T_o} (T_o-t)^2 dt$$

$$= \frac{NE}{2T_o}\left[4T_o^3 - \frac{T_o^3}{3} + \frac{8T_o^3}{3} + 2T_o^3 - 8T_o^3 + T_o^3 + T_o^3 + \frac{T_o^3}{3} - T_o^3\right]$$

$$= \frac{NE}{2T_o}\left[\frac{5T_o^3}{3}\right]$$

$$= \frac{5NET_o^2}{6}$$

APPENDIX D

IMPULSE RESPONSE OF THE EQUIVALENT FILTER FOR THE WD BASED RECEIVER

The impulse response $h_w(t)$ of an equivalent filter which can replace the WD based receiver is the output of the WD based receiver for an input signal $r(t) = \delta(t)$

Using equation (5 20) we get

$$h_w(t) = \int_{\eta=t}^{t+T_o} \int_{\beta=-\infty}^{\infty} \delta(\eta+\frac{\beta}{2}) f_t^*(\eta-\frac{\beta}{2}) e^{-j\{\omega_o+2a(\eta-t)\}\beta} \, d\beta \, d\eta$$

$$= \int_{t}^{t+T_o} 2 \, f_t^*(2\eta) e^{2j\eta\{\omega_o+2a(\eta-t)\}} \, d\eta$$

$$= \int_{t}^{t+T_o} 2 \, . \, f^*(2\eta-t) \, e^{2j\eta\{\omega_o+2a(\eta-t)\}} \, d\eta$$

Now $h_w(t)$ can be computed for different ranges of t

Case 1

$$t \le -2T_o$$

As η goes from t to $(t+T_o)$, $(2\eta-t)$ goes from t to $(t+2T_o)$ Since both t and $(t+2T_o)$ are negative values we get a zero integration value.

Case 2

$$-2T_o \le t \le -T_o$$

Here the limits of η, which will keep $(2\eta-t)$ in the range zero to T_o, are $t/2$ and $(t+T_o)$

$$h_w(t) = \int_t^{t+T_o} 2\sqrt{\frac{E}{T_o}}\, e^{-j\omega_o(2\eta-t)}\, e^{-j\alpha(2\eta-t)^2}\, e^{2j\eta\{\omega_o+2\alpha(\eta-t)\}}\, d\eta$$

$$= \int_{t/2}^{t+T_o} 2\sqrt{\frac{E}{T_o}}\, e^{j\omega_o t}\, e^{-j\alpha t^2}\, d\eta \qquad \text{(D 1)}$$

$$= \sqrt{\frac{E}{T_o}}\,(2T_o+t)\, e^{j\omega_o t}\, e^{-j\alpha t^2}$$

Case 3

$$-T_o \leq \eta \leq 0$$

Here limits of η should be $t/2$ and $(\frac{T_o+t}{2})$

Using equation (D 1) we get

$$h_w(t) = \int_t^{(t_o+t)/2} 2\sqrt{\frac{E}{T_o}}\, e^{j\omega_o t}\, e^{-j\alpha t^2}\, d\eta$$

$$= \sqrt{\frac{E}{T_o}}\; T_o\, e^{j\omega_o t}\, e^{-j\alpha t^2}$$

Case 4

$$0 \leq t \leq T_o$$

Here the required limits on η are t and $(\frac{T_o+t}{2})$

Using equation (D 1) we get

$$h_w(t) = \int_t^{(T_o\ t)/2} 2\sqrt{\frac{E}{T_o}}\, e^{j\omega_o t}\, e^{-j\alpha t^2}$$

$$= \sqrt{\frac{E}{T_o}}\; T_o - t\; e^{j\omega_o t}\, e^{-j\alpha t^2}$$

Case 5

$$T_o \leq t$$

As in case 1 here also there no range of η from t to $(t+T_o)$ which will make $f^*(2\eta - t)$ nonzero Hence $h_w(t) = 0$

APPENDIX E

ENVELOPES OF THE OUTPUTS OF THE CORRELATOR RECEIVER AND THE WD BASED RECEIVER FOR ONE RETURN : SIMULATION RESULTS

APPENDIX F

ENVELOPES OF THE OUTPUTS OF THE CORRELATOR RECEIVER AND THE WD BASED RECEIVER FOR TWO RETURNS : SIMULATION RESULTS

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